Augmented bundles and real structures

Luis Ángel Calvo Pascual

ICMAT-CSIC-UAM-UCM-UC3M

A dissertation submitted in partial fulfilment

of the requirements for the degree of

Doctor in Mathematics

at the Universidad Autónoma de Madrid.

Supervised by: Óscar García Prada.

June, 2018.

Acknowledgements

First of all, I would like to thank Óscar García Prada for giving me the opportunity to research under his supervision. I am very grateful with him for always offering me to make research stays, participate in Workshops, Conferences, Schools and other mathematical events and for having introduced me mathematicians whom I admire such as: M. Atiyah, N. Hitchin, S. Donaldson, P. Newstead, M.S. Narasimhan, C.S. Seshadri, S. Ramanan, D. Mumford, etc. Thanks also for having taught me not only mathematics, but moral and aesthetic values.

Thanks to the various Spanish institutions that supported this research: Ministry of Education, CSIC, ICMAT, UAM. Thanks to the foreign institutions where I made research stays: U. McGill (Montreal), Tata Institute (Mumbay) and CMI (Chennai). Thanks to the mathematicians J. Hurtubise, I. Biswas and V. Balaji for their hospitality and supervision.

Thanks to Oscar's research group and collaborators: Rafael, Luis, Mario, Peter, Steve, André, Tomás, Ignasi, Emilio, Roberto, Ana, Alfonso, David, David, Artur and Manuel. I am very grateful with Emilio, Rohith, Artur and Manuel; thank you for your help and the nice experiences that we have lived together. Thanks to F. Schaffhauser for his help in the beginning.

Finally, I want to thank my family and friends for their unconditional support.

Contents

0.1 Introduction

Real structures on a mathematical object equipped with a holomorphic structure are antiholomorphic involutions. Klein surfaces, that is, pairs consisting of a Riemann surface and a real structure, were first studied by Klein and Weichold, who gave in [54] and [86] a topological classification of them. Real structures on complex Lie algebras and complex Lie groups were studied by E. Cartan, Tits, Vogan, etc. Real structures on complex vector spaces are antilinear involutions. Real structures for vector bundles are antiholomorphic involutions, which are antilinear in the fibres. This notion was introduced by Atiyah in [4]. Gross and Harris gave in [43] a classification of real line bundles on Klein surfaces. A common feature of these involutions is that their fixed-point subspaces are defined over the real numbers. For instance, they are real algebraic curves in the case of Klein surfaces, real vector spaces in the case of real structures on complex vector spaces, or a real Lie groups in the case of real structures on complex Lie groups.

Vector bundles or principal bundles equipped with various extra structures are called generically as **augmented bundles**. This thesis deals with the study of three examples of augmented bundles (Higgs pairs, G-Higgs bundles and G-parabolic Higgs bundles, where G is a real reductive Lie group) in the presence of real structures on their constituent components.

0.1.1 Historical background

The Hodge correspondence establishes an isomorphism between representations $\mathrm{Hom}(\pi_1(X), G)$ of the fundamental group $\pi_1(X)$ of a compact Riemann surface X into an abelian group G and the cohomological group $H^1(X, G)$, that parametrizes isomorphism classes of G-bundles over X. The augmented bundles that we study have appeared as sucessive generalizations of the **non-abelian Hodge correspondence**, that is an extension, when the group is not abelian, of the Hodge correspondence.

If G is the unitary group $\mathrm{U}(n)$, the non-abelian Hodge correspondence is known as the theorem of Narasimhan and Seshadri. The moduli space of stable holomorphic vector bundles of rank n and degree d on a compact Riemman surface X was constructed by Mumford in [58]. Narasimhan and Seshadri proved in [57] that this moduli space, for $d = 0$ and genus g of X greater than one, was homeomorphic to the moduli space of irreducible representations of $\pi_1(X)$ into $\mathrm{U}(n)$. Seventeen years later, Donaldson gave in [30] a new proof of the theorem of Narasimhan and Seshadri using the approach of Gauge Theory.

If G is a compact Lie group, the correspondence was proved by Ramanathan. He introduced in [68] and [69] a notion of stability for G-bundles and constructed the moduli space $M_d(G)$ of G-bundles of topological class $d \in \pi_1(G)$, for any compact Riemann surface X of genus $g \geq 2$. Ramanathan proved in [67] that $M_0(G)$ is homeomorphic to the moduli space of representations $R(G)$ of $\pi_1(X)$ into G.

If G is a non compact reductive Lie group, Higgs bundles are required in the correspondence. They were first introduced by Hitchin in [46], for $G = \mathrm{SL}(2, \mathbb{C})$. He constructed its moduli space $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}))$ relating Mumford's notion of stability with solutions of the Hitchin equations. Hitchin pointed out that a solution of the Hitchin equations produces a reductive flat connection. The existence of a harmonic metric on a reductive flat bundle was proved by Donaldson in [32], for $\mathrm{SL}(2, \mathbb{C})$ and by Corlette in [29], for a complex reductive Lie group G. The **Corlette-Donaldson correspondence** establishes a homeomorphism between the moduli space of reductive flat connections and the moduli space of harmonic bundles, which, in turn, is in bijection with solutions of the Hitchin equations. The existence of the moduli space of G-Higgs bundles $\mathcal{M}(G)$, for any complex reductive Lie group G, was constructed by Simpson in [79, 80]. He generalized the non-abelian Hodge correspondence in [78] for G-Higgs bundles over compact Kähler manifolds.

Gothen, Garcia-Prada and Mundet i Riera introduced in [39] the notion of **Higgs pairs** (E, φ), which consist of a holomorphic G-bundle E over a compact Riemann surface X, where G is a connected reductive complex Lie group and a holomorphic section φ of the vector bundle associated to E, via a representation $\rho : G \to \mathrm{GL}(\mathbb{V})$ of G on a complex vector space $\mathbb{V}$, tensored by a line bundle L. Let H be a maximal compact subgroup of G, such that ρ reduce to a representation $\rho : H \to \mathrm{U}(\mathbb{V})$. These authors defined α-polystability, where α is an element of the center of the Lie algebra $\mathfrak{h}$ of H and proved the **Hitchin-Kobayashi correspondence**: Fix a bi-invariant metric B on $\mathfrak{h}$, an H-invariant Hermitian product $\langle\, ,\, \rangle$ on $\mathbb{V}$, a Kähler form ω of X and a Hermitian metric h_L on L. A Higgs pair (E, φ) is α-polystable if and only if there is reduction of structure group of E from G to H satisfying the **Hermite-Einstein-Higgs equation**

$$\Lambda F_h + \mu_h(\varphi) = -\sqrt{-1}\,\alpha\,, \tag{0.1.2}$$

where F_h is the curvature of the unique connection on the reduced bundle E_H compatible with the holomorphic structure of E, Λ denotes the contraction with ω and μ_h is defined as follows: let $Q : E_H(\mathfrak{u}(\mathbb{V}))^* \to E_H(\mathfrak{h})$ be the map induced by $d(\rho)^* : \mathfrak{u}(\mathbb{V})^* \to \mathfrak{h}^*$ and $B : \mathfrak{h} \to \mathfrak{h}^*$, then $\mu_h(\varphi) := Q(-\frac{\mathrm{i}}{2}\varphi \otimes \varphi^*)$, with φ^* being the image of φ by the metric on $E(\mathbb{V}) \otimes L$ induced by h, the hermitian product $\langle\, ,\, \rangle$ and h_L.

If G is a reductive real Lie group, G**-Higgs bundles** are K-Higgs pairs, where K is the canonical bundle and ρ is the isotropy representation $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ being the Cartan decomposition of the Lie algebra of G. The existence of the moduli space of G-Higgs bundles follows from the general construction of Schmitt [73]. The non-abelian Hodge correspondence is a consequence of the Hitchin-Kobayashi correspondence and the Corlette-Donaldson correspondence.

If X is an elliptic curve, after Atiyah's identification of the moduli space of vector bundles of rank n over X with the n-fold symmetric product $\mathrm{Sym}^n(X)$ of X, Lazlo, Friedman, Morgan and Witten in [35] and [36] constructed a bijective morphism from $(X \otimes_{\mathbb{Z}} \Lambda_T)/W$ to the moduli of G-bundles $M(G)$ over X, where T is a Cartan subgroup of a complex reductive Lie group G, Λ_T is the cocharacter lattice and $W = N_G(T)/T$ is the associated Weyl group. Since $M(G)$ is normal, this morphism is an isomorphism by Zariski's main Theorem. Thaddeus described in [81] the normalization $\overline{\mathcal{M}}(G)$ of $\mathcal{M}(G)$ as

$$\overline{\mathcal{M}}(G) \;\cong\; (T^*X \otimes_{\mathbb{Z}} \Lambda_T)/W\,.$$

Franco, Garcia-Prada and Newstead identified in [33] and [34], the reduced moduli space of G-Higgs bundles $\mathcal{M}(G)_{\mathrm{red}}$ with $(T^*X \otimes_{\mathbb{Z}} \Lambda_T)/W$, The reduced moduli space of representations of the fundamental group into G is isomorphic to $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}}$

Λ_T/W and the non-abelian Hodge correspondence is induced by the diffeomorphism between $T^*\widehat{X}$ and $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ given by the Hodge correspondence.

If X is a Riemann surface and we fix some points on it, reductive representations into a reductive Lie group G with prescribed holonomy around those points are in one-to-one correspondence with polystable parabolic bundles. The non-abelian Hodge correspondence was firstly studied by Metha and Seshadri in [59], for $G = \mathrm{U}(n)$. Bhosle and Ramanathan in [9], Seshadri and Balaji in [6] and Teleman and Woodward in [83] extended this correspondence for compact Lie groups, considering rational holonomies around the points. When G is not compact, parabolic Higgs bundles were introduced by Simpson in [76], for $G = \mathrm{GL}(n, \mathbb{C})$. He constructed the moduli space of filtered parabolic local systems in [77]. If we consider rational holonomies, the moduli space of representations is in one-to-one correspondence with parahoric torsors, introduced by Pappas and Rapoport in [66]. Balaji and Seshadri constructed in [6] the moduli space and proved that there is a bijection between the moduli space of parahoric torsors and the moduli space of Γ-equivariant bundles over a cyclic cover Y of X ramified over each prescribed point of X, where $Y \to X$ is a Galois covering of group Γ. Biquard, Garcia-Prada and Mundet i Riera introduced in [22] **parabolic G-Higgs bundles**, where G is a real reductive Lie group. They proved a non-abelian Hodge correspondence with arbitrary holomoy arround the punctures and with reductions near the points having prescribed asymptotic behaviour. The existence of the moduli space of parabolic G-Higgs bundles follows from the general construction of Schmitt [73].

The theory of real vector bundles over Klein surfaces was developed between the years 2010-2012. Schaffhauser in [71] and [72] and Schaffhauser and Liu in [56] described explicitly the moduli spaces of these bundles. Biswas, Huisman and Hurtubise in [17] introduced **pseudo-real G-bundles** over real Kähler manifolds, where G is a complex reductive Lie group. They defined Behrend's stability for these objects and established a non-abelian Hodge correspondence between the moduli space of pseudo-real G-bundles and the moduli space of representations of the orbifold fundamental group. Biswas, Garcia-Prada and Hurtubise extended the correspondence in [15] and [16] for pseudo-real G-Higgs bundles. Biswas and Garcia-Prada described in [14] how the moduli space of pseudo-real G-Higgs bundles over X appears naturally as the fixed point set of antiholomorphic involutions in the moduli space of G-Higgs bundles, depending on real structures on X and G. In branes language, the moduli space of pseudo-real G-Higgs bundles is an (A, B, A)-brane. The (A, A, B)-brane case was studied by Schaposnik and Baraglia in [7] and Biswas and Garcia-Prada in [14].

0.1.2 Main results

This thesis is concerned with the study of some augmented bundles equipped with real structures. More specifically, we study: real Higgs pairs, real G-Higgs bundles and real parabolic G-Higgs bundles, where G is a real form of a semisimple group $G^{\mathbb{C}}$. These bundles generalize the notion of pseudo-real Higgs bundles given in [17]. We prove that they appear naturally as fixed points of involutions in the moduli space of Higgs pairs, G-Higgs bundles and parabolic G-Higgs bundles, respectively. We also prove a non-abelian Hodge for real G-Higgs bundles and real parabolic G-Higgs bundles.

To obtain these results, we first introduce $(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)$-**real (quaternionic) Higgs pairs** (E, φ, σ_E) over a compact Klein surface (X, σ_X), that depend on: a real structure σ_G on a complex reductive Lie group G, a real (quaternionic) structure σ_L on a line bundle L, a real (quaternionic) structure $\sigma_{\mathbb{V}}$ on a complex vector space $\mathbb{V}$, an element c in the center Z of G invariant under σ_G of order two and a parameter $\pm$. We define the stability of Behrend, considering only reductions such that the adjoint of the reduced bundle is preserved by $\sigma_{\mathrm{Ad}(E)}$, that is an antiholomorphic involution in the adjoint bundle $\mathrm{Ad}(E)$ induced by σ_E and σ_G.

A σ_E-**compatible Hermite-Einstein-Higgs reduction** of a $(\sigma_X, \sigma_G, c, \sigma_L, \pm)$-real (quaternionic) Higgs pair (E, φ, σ_E) is a reduction h of structure group of E from G to a maximal compact subgroup H of G, such that the reduced bundle E_H is preserved by σ_E and h satisfies Equation (0.1.2).

Theorem 2.5.1 [Hitchin-Kobayashi correspondence] A $(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)$-real (quaternionic) Higgs pair (E, φ, σ_E) is α-polystable if and only if there exists a

σ_E-compatible Hermite-Einstein-Higgs reduction of structure group of E from G to H.

$(\sigma_X, \sigma_G, c, \alpha, \pm)$**-real** G**-Higgs bundles** are a particular case of real Higgs pairs, where G is a semisimple real form of a connected reductive Lie group $G^{\mathbb{C}}$ equipped with a conjugation μ of $G^{\mathbb{C}}$ such that $(G^{\mathbb{C}})^{\mu} = G$ and a conjugation τ of $G^{\mathbb{C}}$ such that $(G^{\mathbb{C}})^{\tau} = H^{\mathbb{C}}$ is a maximal compact subgroup of $G^{\mathbb{C}}$. In this case, σ_G is a (μ, τ)-compatible conjugation of a real form (Subsection 1.3.3) and c is an element of order two, invariant under σ_G, that lives in the intersection of the centers of $H^{\mathbb{C}}$ and $G^{\mathbb{C}}$. We denote by $\mathcal{M}(\sigma_X, \sigma_G, c, \pm)$ the moduli space of real G-Higgs bundles and by $\mathcal{R}(\sigma_X, \sigma_G, c, \pm)$ the moduli space of reductive $(\sigma_X, \sigma_G, c, \pm)$-compatible representations of the orbifold fundamental group of X.

Theorem 3.4.6 [Non-abelian Hodge correspondence] There is bijective correspondence between $\mathcal{M}(\sigma_X, \sigma_G, c, \pm)$ and $\mathcal{R}(\sigma_X, \sigma_G, c, \pm)$.

Consider the following involutions in the moduli space of G-Higgs bundles $\mathcal{M}$

$$\begin{array}{rccc} i_{\mathcal{M}}(\sigma_X, \sigma_G)^{\pm}: & \mathcal{M} & \longrightarrow & \mathcal{M} \\ & (E, \varphi) & \longmapsto & (\sigma_X^* \sigma_G E, \pm \sigma_X^* \sigma_G \varphi)\,. \end{array} \tag{0.1.3}$$

Thanks to Theorem 2.5.1, there is an embedding of $\mathcal{M}(\sigma_X, \sigma_G, c, \pm)$ in $\mathcal{M}$. We denote by $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \pm)$ the image of the embedding, and by $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \pm)_{\mathrm{sm}}$ the intersection of $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \pm)$ and the smooth locus $\mathcal{M}_{\mathrm{sm}}$ of $\mathcal{M}$.

Theorem 3.6.1 Fixed points of $i_{\mathcal{M}(\alpha)}(\sigma_X, \sigma_G)^{\pm}$ and the moduli space of real polystable G-Higgs bundles are related by:

(1)
$$\mathcal{M}^{i_{\mathcal{M}}(\sigma_X, \sigma_G)^{\pm}} \supseteq \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \pm)\,.$$

(2) For $g(X) \geq 2$ and if we restrict to the smooth locus sm, then
$$\mathcal{M}_{\mathrm{sm}}^{i_{\mathcal{M}}(\sigma_X, \sigma_G)^{\pm}} \cong \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \pm)_{\mathrm{sm}}\,.$$

Thanks to Theorem 3.4.6, we prove a similar result for the moduli space of representations of the fundamental group.

Let $\mathcal{M}(Z)$ be the moduli space of topologically trivial Z–Higgs bundles over an elliptic curve, where Z is the center of G. $\mathcal{M}(Z)$ is a complex group. Let $\mathcal{F} = (F, \phi) \in \mathcal{M}(Z)$ be such that $\mathcal{F} = \mathcal{F}^{-1}$; we define the following involution in the moduli space $\mathcal{M}$ of G-Higgs bundles over an **elliptic curve**, where G is a reductive complex Lie group

$$\begin{array}{rccc} i_{\mathcal{M}}(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F}) : & \mathcal{M} & \longrightarrow & \mathcal{M} \\ & (E, \varphi) & \longmapsto & \mathcal{F} \otimes \iota_{\mathcal{M}}(\alpha_\epsilon, \sigma_\epsilon, \pm)(E, \varphi) \, , \end{array}$$

where α_ϵ and σ_ϵ are holomorphic or antiholomorphic involutions on X and G, respectively, and the parameter ϵ indicates whether it is the holomorphic case ($\epsilon = +$) or the anti-holomorphic case ($\epsilon = -$). Consider the involution

$$\begin{array}{rccc} \dot{I}(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F}) : & T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T & \longrightarrow & T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \\ & \sum (L_i, \psi_i) \otimes \lambda_i & \longmapsto & [\mathcal{F}] + \sum i_{\mathcal{M}}(\alpha_\epsilon, \alpha_\epsilon, \pm)(L_i, \psi_i) \otimes \sigma_\epsilon(\lambda_i) \, . \end{array}$$

In Corollary 4.5.2, we describe explicitly the fixed points of the involutions:

$$\mathcal{M}^{\iota_{\mathcal{M}}(t_y \circ \alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F})} = \bigcup_{\overline{\omega} \in W/_{\sigma_\epsilon} W} \left(T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T\right)^{\omega\left(\dot{I}(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F})\right)} / N_W(T^{\omega\sigma_\epsilon}) \, .$$

We prove a similar result for the moduli space of representations of the fundamental group. Using these results we describe explicitly the moduli space of $(\sigma_X, \sigma_G, c, \pm)$-real Higgs bundles.

Let σ_G be a (μ, τ)-compatible conjugation of a real form G of $G^{\mathbb{C}}$. Let $S = \{x_1, \ldots, x_r\}$ be a finite set of points of a compact Klein surface (X, σ_X), such that $\sigma_X(S) = S$. Let $\mathcal{A}$ be an invariant alcove under σ_G. Let α be a (S, σ_G)-compatible set (Definition 5.2.2) of weights in $\sqrt{-1}\mathcal{A}$. A **$(\sigma_X, \sigma_G, c, \alpha, \pm)$-real parabolic G-Higgs bundles** is a $(\sigma_X, \sigma_G, c, \pm)$-real G-Higgs bundle such that the parabolic structures of weights α over a set of points S are $(S, \sigma_{\mathrm{Ad}(E)})$-compatible and the Higgs field is invariant under the involution $P\mathrm{Ad}(f)$ induced by σ_G. We denote by $\mathcal{M}(\sigma_X, \sigma_G, \alpha, \pm, \mathcal{L})$ the moduli space of real meromorphically equivalent classes of $(\sigma_X, \sigma_G, c, \alpha, \pm)$-real polystable parabolic Higgs bundles such that the projection $\mathrm{Gr}\,\mathrm{Res}_{x_i} \varphi$ of the residue of φ at x_i is $\mathcal{L}_i$ and $\mathcal{L} := (\mathcal{L}_1, \ldots, \mathcal{L}_r)$ is $(S, \mathrm{Ad}(f))$-compatible. A **$(\sigma_X, \sigma_G, c, \beta, \pm)$-compatible filtered parabolic local system** is a triple (F, S, σ_F) such that $F_{X \setminus S}$ is a local system F, equipped with a $(\sigma_X, \sigma_G, c, \pm)$-compatible structure (Definition 3.3.2) and for every x_i and every ray ρ_i going to x_i

we have to choose a pair (P_i,χ_i) where P_i is a parabolic subgroup of $F(G)|_{\rho_i}$ isomorphic to P_{β_i}, P_i is invariant under the monodromy transformation around x_i, σ_F and σ_G induce an involution $\sigma_{F(G)}$ on $F(G)$ such that $(P_1,\ldots,P_r)\in\prod_{i=1}^r F(G)|_{\rho_i}$ is a $(S,\sigma_{F(G)})$-compatible element and χ_i is a stricly antidominant character of P_i. We denote by $\mathcal{S}(\sigma_X,\sigma_G,c,\beta,\pm,\mathcal{C})$ the moduli space of real filtered local systems of weight β with monodromies $\mathcal{C}=(\mathcal{C}_1,\cdots,\mathcal{C}_r)$, with $\mathcal{C}$ being a (S,σ_G)-compatible element. Let $\mathcal{R}(\sigma_X,\sigma_G,c,\pm,\mathcal{C})$ be the moduli space of reductive representations of the orbifold fundamental group of $X\backslash S$ into G, such that the restriction to $\pi_1(X\backslash S)$ has as images of the loops arround the prescribed points the conjugacy classes $\mathcal{C}\in\prod_{i=1}^r\mathrm{Conj}(G)$, with $\mathcal{C}$ being a (S,σ_G)-compatible element.

Theorem 5.7.3 [Non-abelian Hodge correspondence] If the weights and the monodromies are related by Table 5.1, then:

(1) $\mathcal{M}(\sigma_X,\sigma_G,c,\alpha,\pm,\mathcal{L})$ is homeomorphic to $\mathcal{S}(\sigma_X,\sigma_G,c,\beta,\pm,\mathcal{C})$.

(2) $\mathcal{S}(\sigma_X,\sigma_G,c,0,\pm,\mathcal{C})$ is homeomorphic to $\mathcal{R}(\sigma_X,\sigma_G,c,\pm,\mathcal{C})$.

The involutions $i_{\mathcal{M}}(\sigma_X,\sigma_G)^\pm$ defined in (0.1.3) can be extended to the moduli space $\mathcal{M}(\alpha,\mathcal{L})$. We denote by $i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X,\sigma_G)^\pm$ the extended involutions.

Theorem 5.8.1 The fixed points of the involution $i_{\mathcal{M}(\alpha)}(\sigma_X,\sigma_G)^\pm$ and the moduli space of $(\sigma_X,\sigma_G,c,\alpha,\pm)$-real parabolic G-Higgs bundles of class $\mathcal{L}$ are related by:

(1)

$$\mathcal{M}(\alpha,\mathcal{L})^{i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X,\sigma_G)^\pm}\supseteq\bigcup_{c\in Z(H^{\mathbb{C}})_2^{\sigma_G}\cap Z(G^{\mathbb{C}})}\widetilde{\mathcal{M}}(\sigma_X,\sigma_G,c,\alpha,\pm,\mathcal{L})\,.$$

(2) For $g(X)\geq 2$ and if we restrict to the smooth locus sm, then

$$\mathcal{M}(\alpha,\mathcal{L})_{\mathrm{sm}}^{i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X,\sigma_G)^\pm}\cong\bigcup_{c\in Z(H^{\mathbb{C}})_2^{\sigma_G}\cap Z(G^{\mathbb{C}})}\widetilde{\mathcal{M}}(\sigma_X,\sigma_G,c,\alpha,\pm,\mathcal{L})_{\mathrm{sm}}\,.$$

Thanks to Theorem 5.7.3 we prove similar results for the moduli space of filtered parabolic local systems and for the moduli space of representations of the fundamental group of $X\backslash S$ into G.

0.1.3 Contents

Chapter 1 is devoted to introduce real structures for the main objects studied in this thesis. In Section 1.1, we recall the definition and topological classification of compact Klein surfaces. In Subsection 1.1.1, we state the classification of real elliptic curves and describe actions of real structures on $\pi_1(X)$. In Subsection 1.1.2, we define real Kähler forms, and we prove that if a Klein surface (X, σ_X) is equipped with a Kähler form, then it admits a σ_X-real Kähler form. In Section 1.2, we define real, quaternionic and real unitary structures on complex vector spaces.

Section 1.3 is divided in three parts; in the first one, we define real structures on Lie algebras, and we recall their relation with automorphisms of order two, established by Cartan. In the second part, we study real structures on Lie groups and their relation with holomorphic automorphisms of the group (Cartan's Theorem). As a consequence, given a real structure σ_G on a group G, there exists a maximal compact subgroup of G invariant under σ_G. In the third part, we define (μ, τ)-compatible conjugations of $G^{\mathbb{C}}$ and we study its basic properties.

In Section 1.4, we define real actions of a group on a manifold, both equipped with real structures. A particular example are real linear representations. For subsequent use in this thesis, we also define quaternionic and real unitary representations. In Section 1.5, we define real, quaternionic and real unitary structures on a vector bundle E. In Subsection 1.5.1, we reformulate these definitions in terms of the Dolbeaut operator of the underlying C^∞-bundle $\mathbb{E}$. Compatible Hermitian metrics on $\mathbb{E}$ are defined in Subsection 1.5.2. In Subsection 1.5.3, we define compatible conections on $\mathbb{E}$ and we prove that if a compatible Hermitian metric on $\mathbb{E}$ is fixed, then there is a bijective correspondence between real, quaternionic and real unitary Dolbeaut operators and compatible connections on $\mathbb{E}$. In Subsection 1.5.4, we recall the topological classification of real, quaternionic and real unitary vector bundles over a Klein surface following [17, p.8].

Section 1.6 deals with real G-bundles, where G is a reductive complex Lie group. In Subsection 1.6.1, we define real G-bundles in terms of the holomorphic structure of the underlying C^∞-bundle $\mathbb{E}$. Compatible reductions and conections are defined in Subsection 1.6.2 and we prove that if a compatible reduction on $\mathbb{E}$ is fixed, then there is a bijective correspondence between real holomorphic structures and compatible connections. In Subsection 1.6.3 we recall the topological classification of pseudo-real G bundles given in [17, p.13]. We study real structures on associated bundles and we give some examples in Section 1.7.

In Chapter 2, we generalize the work on Higgs pairs contained in [39], [61] and [25] to the case in which all objects are equipped with a real structure. First, we introduce real and quaternionic structures on Higgs pairs and we give some examples in Section 2.1. In Section 2.2, we recall the Hermite-Einstein-Higgs equation. We include Behrend's stability of real (quaternionic) Higgs pairs in Section 2.3. In Section 2.4, we recall from [62, Section 2.3.3] that Higgs pairs can be considered as an infinite dimensional symplectic manifold such that the action of the gauge group is hamiltonian, the moment map is the Hermite-Einstein-Higgs equation and the degree of a L-Higgs pair can be calculated as a maximal weight.

We state the Hitchin-Kobayashi correspondence for real (quaternionic) Higgs pairs in Section 2.5. In Section 2.6, we define the semisimple infinitesimal automorphisms space of a real (quaternionic) Higgs pair and we prove that a real (quaternionic) Higgs pair is stable if and only if the infinitesimal automorphisms are central. In Section 2.7, we prove that a polystable real (quaternionic) Higgs pair admits a Jordan-Hölder reduction and the result is a stable real (quaternionic) Higgs pair.

In Section 2.8, we prove that stability in real (quaternionic) simple Higgs pairs implies existence of solutions of the Hermite-Einstein-Higgs equation, following the ideas in [61]. More concretely, first we define real (quaternionic) simplified Higgs pairs in Subsection 2.8.1. Real (quaternionic) L-Higgs pairs can be considered as real (quaternionic) simplified Higgs pairs. We define stability for real (quaternionic) simplified Higgs pairs. In Subsection 2.8.2, we adapt the symplectic point of view of Section 2.4 for simplified Higgs pairs. In Subsection 2.8.3, we state our problem in terms of finding a metric on which the integral of the moment map attains a minimum. We prove the statement in Subsection 2.8.4, replicating the proof in [61].

In Section 2.9, we prove that existence of solutions of the Hermite-Einstein-Higgs equation implies polystability of the real (quaternionic) Higgs pair and in Section 2.10, we identify real Higgs pairs as fixed points of involutions defined on Higgs pairs.

A non-abelian Hodge correspondence for real G-Higgs bundles, where G is a real reductive Lie group, is proved in Chapter 3. We start defining real G-Higgs bundles as a particular case of real Higgs pairs in Section 3.1. By Theorem 2.5.1, there is a bijective correspondence between the moduli space of real G-Higgs bundles and the moduli space $\mathcal{A}_0$ of triples consisting of: compatible conections, Higgs sections and real structures on the underlying C^∞-bundle satisfying the Hitchin equations. This is done in Section 3.2. In Section 3.3, we prove a bijective correspondence between the moduli space of compatible local systems and the moduli space of representations of the orbifold fundamental group. The Donaldson-Corlette correspondence for compatible local systems $\mathcal{S}$ is proved in Section 3.4. The correspondence establishes a bijection between $\mathcal{S}$ and $\mathcal{A}_0$. This result together with the theorems of Section 3.2 and Section 3.3 proves the non-abelian Hodge correspondence. In Section 3.5 we define involutions in the moduli space of G-Higgs bundles, the moduli space of local systems, and the moduli space of representations of the fundamental group. Thanks to the non-abelian Hodge correspondence for real G-Higgs bundles and Section 2.10, we obtain a description of the fixed point locus of the involutions in Section 3.6

Chapter 4 is dedicated to the study of real G-Higgs bundles over real elliptic curves, where G is a complex reductive Lie group. First, we review in Section 4.1 the description of the moduli space of G-Higgs bundles over an elliptic curve X and the moduli space of representations of $\pi_1(X)$ into G. We define, in Section 4.2, involutions $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$, $J(\alpha_\epsilon$ and $\sigma_\epsilon, \pm)$ on these moduli spaces, respectively. In Section 4.3, we describe explicitly these involutions in the case of Higgs line bundles. Using that, we obtain an explicit description of the involutions for the general case in Section 4.4. We describe explicitly their fixed point loci in Section 4.5 and the moduli space of real G-Higgs bundles over X, in Section 4.6.

A Non-abelian Hodge correspondence for real parabolic G-Higgs bundles, where G is a real reductive Lie group, is proved in Chapter 5. We identify these moduli spaces as fixed points of involutions. In Section 5.1, we do an overview of parabolic G-Higgs bundles, for G real reductive Lie group, following [22] as our main reference. First, we review, in Subsection 5.1.1, the definitions of the moduli space of parabolic

G-Higgs bundles; in Subsection 5.1.2, the moduli space of parabolic G-local systems; and in Subsection 5.1.3, the moduli space of representations of $\pi_1(X\backslash S)$. These moduli spaces depend on parabolic weights, residues, monodromies and conjugacy classes of G. We conclude this overview in Subsection 5.1.4, where we recall that these moduli spaces are in one-to-one correspondence choosing suitable classes. We define in Section 5.2 involutions on these moduli spaces depending on a (μ, τ)-compatible conjugation σ_G of $G^{\mathbb{C}}$ and a real structure σ_X on X. We introduce the definition of real parabolic G-bundles in Section 5.3 and we define Behrend's stability in Section 5.4. We prove a Hitchin-Kobayashi correspondence (Theorem 5.5.5) for real parabolic G-bundles in Section 5.5. In Section 5.6, we define real parabolic filtered local systems and also notions of Behrend z-stability for these objects. The Hitchin-Simpson correspondence for real parabolic filtered local systems (Theorem 5.7.1) is proved in Section 5.7. Thanks to the Hitchin-Kobayashi correspondence and the Hitchin-Simpson correspondence (Theorem 5.7.1), there are embeddings of these moduli spaces of real objects in the moduli space of parabolic G-Higgs bundles and the moduli space of parabolic G-local systems. Finally, we prove in Section 5.8, that the set of fixed points of the involutions are, precisely, the images of these embeddings.

The Appendix A is divided in two parts. In the first part, Section A.1, following [44, 50, 53], we compute the relative position of two simpleces with respect to two equivalent spherical Tits buildings: the one associated to G and the one associated to the boundary at infinity $\partial_\infty(H\backslash G)$ of the quotient group $H\backslash G$. In the second part, Section A.2, we give two definitions of parabolic degree. The parabolic degree of Biquard, Garcia-Prada and Mundet i Riera (see [22]) is the sum of the usual degree and the relative degree of two simpleces on $\partial_\infty(H\backslash G)$. Teleman and Woodward's parabolic degree (see [83]) is the sum of the usual degree and the relative degree of two simpleces of the spherical Tits building determined by G. We prove that these two definitions are equivalent.

0.2 Introducción

Las estructuras reales en un objeto matemático equipados con una estructura holomorfa son involuciones antiholomorfas. Las superficies de Klein, esto es, pares que consisten en una superficie de Riemann y una estructura real, fueron estudiadas por primera vez por Klein y Weichold, quienes dieron una clasificación topológica de las mismas a finales del siglo XIX. Las estructuras reales en álgebras de Lie complejas y grupos de Lie complejos fueron estudiadas por E. Cartan, Tits, Vogan, etc. Las estructuras reales en espacios vectoriales complejos son involuciones antilineales. Las estructuras reales en fibrados vectoriales son involuciones antiholomorfas, que son antilineales en las fibras. Esta noción fue introducida por Atiyah in [4]. Gross y Harris dieron in [43] una clasificación de fibrados de línea reales en superficies Klein. Una característica común de todas estas estructuras reales es que son involuciones cuyos subespacios de puntos fijos están definidos sobre los números reales. Por ejemplo, son curvas algebraicas reales en el caso de superficies de Klein, son espacios vectoriales reales en el caso de estructuras reales sobre espacios vectoriales complejos o grupos de Lie reales en el caso de estructuras reales sobre grupos de Lie complejos.

Los fibrados vectoriales o fibrados principales equipados con varias estructuras adicionales se designan genéricamente como **fibrados aumentados**. Esta tesis trata del estudio de tres ejemplos de fibrados aumentados (fibrados de Higgs, pares de Higgs y fibrados de Higgs parabólicos) en presencia de estructuras reales en sus componentes constituyentes.

0.2.1 Antecedentes históricos

La correspondencia de Hodge establece un isomorfismo entre el grupo de representaciones $\mathrm{Hom}(\pi_1(X), G)$ del grupo fundamental $\pi_1(X)$ de una superficie de Riemann compacta X en un grupo de Lie abeliano G y el grupo cohomológico $H^1(X, G)$, el cual clasifica G-fibrados en X. Los fibrados aumentados que estudiamos han aparecido como generalizaciones sucesivas de la **correspondencia de Hodge no-abeliana**, que es una extensión, cuando el grupo no es abeliano, de la correspondencia de Hodge.

Si G es el grupo unitario $\mathrm{U}(n)$, la correspondencia correspondencia de Hodge no-abeliana se conoce como el teorema de Narasimhan y Seshadri. El espacio de moduli de fibrados vectoriales holomorfos estables de rango n y grado d en una superficie de Riemman compacta X fue construida por Mumford en [58]. Narasimhan y Seshadri demostraron en [57] que este espacio de moduli, para $d = 0$ y género g de X mayor que uno, era homeomorfo al espacio de moduli de representaciones irreducibles de $\pi_1(X)$ en $\mathrm{U}(n)$. Diecisiete años más tarde, Donaldson dio en [30] una nueva prueba del teorema de Narasimhan y Seshadri utilizando el enfoque de la teoría Gauge.

Si G es un grupo de Lie compacto, la correspondencia fue probada por Ramanathan. Dicho autor introdujo en [68] y [69] una noción de estabilidad para G-fibrados y construyó el espacio de moduli $M_d(G)$ de G-fibrados de clase topológica $d \in \pi_1(G)$, para cualquier superficie de Riemann compacta X de género $g \geq 2$. Ramanathan demostró en [67] que $M_0(G)$ es homeomorfo al espacio de moduli de representaciones $R(G)$ de $\pi_1(X)$ en G.

Si G es un grupo de Lie reductivo no compacto, en la correspondencia se requieren fibrados de Higgs. Fueron por primera vez introducidos por Hitchin en [46], para $G = \mathrm{SL}(2, \mathbb{C})$. Él construyó el espacio de moduli $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}))$ relacionando la noción de estabilidad de Mumford con soluciones a las ecuaciones de Hitchin. Éste autor señaló que una solución de la ecuaciones de Hitchin produce una conexión plana reductiva.

La existencia de una métrica armónica en un fibrado plano reductivo fue probada por Donaldson en [32] para $\mathrm{SL}(2, \mathbb{C})$ y por Corlette en [29] para cualquier grupo de Lie real reductivo G. La **correspondencia de Corlette-Donaldson** establece un homeomorfismo entre el espacio de moduli de conexiones planas reductivas y el espacio de moduli de fibrados armónicos que está en biyección con soluciones de las ecuaciones de Hitchin. La existencia del espacio de moduli de G-fibrados de Higgs $\mathcal{M}(G)$, para cualquier grupo de Lie reductivo complejo G, fue construido por Simpson en [79, 80]. Él generalizó la correspondencia de Hodge no abeliana en [78] para G-fibrados de Higgs sobre variedades Kähler compactas.

Gothen, Garcia-Prada y Mundet i Riera introdujeron en [39] la noción de **par de Higgs** (E, φ) que consiste en un G-fibrado holomorfo E sobre una superficie de

Riemann compacta X, siendo G un grupo de Lie de conexo complejo reductivo y una sección holomorfa φ del fibrado vectorial asociado a E via una representación $\rho : G \to \mathrm{GL}(\mathbb{V})$ de G en un espacio vectorial complejo $\mathbb{V}$ tensorializado por un fibrado de línea L. Sea H un subgrupo compacto maximal de G tal que ρ reduce a una representación $\rho : H \to \mathrm{U}(\mathbb{V})$. Estos autores α-poliestabilidad, donde α es un elemento del centro del álgebra de Lie $\mathcal{H}$ de H y probaron la **correspondencia de Hitchin-Kobayashi**: fijamos una métrica bi-invariante B en $\mathfrak{h}$, un producto hermítico invariante $\langle\ ,\, ,\ \rangle$ en $\mathbb{V}$, una forma K ähler ω de X y una métrica hermítica h_L en L. Un par de Higgs (E, φ) es α-polystable si y sólo si hay una reducción del grupo de estructura de E de G a H que satisface la **ecuación de Hermite-Einstein-Higgs**

$$\Lambda(F_h) + \mu_h(\varphi) = -\sqrt{-1}\,\alpha, \tag{0.2.5}$$

donde F_h es la curvatura de la única conexión del fibrado reducido E_H que es compatible con la estructura holomorfa de E, Λ denota la contracción con ω and μ_h se define de la siguiente manera: sea $Q : E_H(\mathrm{U}(\mathbb{V}))^* \to E_H(\mathfrak{h})$ la aplicación inducida por $d(\rho)^* : \mathfrak{u}(\mathbb{V})^* \to \mathfrak{h}^*$ y $B : \mathfrak{h} \to \mathfrak{h}^*$, entonces $\mu_h(\varphi) := Q(-\frac{\mathrm{i}}{2}\varphi \otimes \varphi^*)$, siendo φ^* la imagen de φ por la metrica en $E(\mathbb{V}) \otimes L$ inducida por h, el producto hermítico $\langle\ ,\ \rangle$ y h_L.

Si G es un grupo de Lie real reductivo, los **G-fibrados de Higgs** son K-pares de Higgs, donde K es fibrado canónico y ρ es la representación de isotropía $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$, siendo $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ la descomposición de Cartan del álgebra de Lie de G. La existencia del espacio de moduli de G-fibrados de Higgs es resultado de la construcción general de Schmitt [73]. La correspondencia no abeliana de Hodge es una consecuencia de la correspondencia de Hitchin-Kobayashi y la correspondencia Corlette-Donaldson.

Si X es una curva elíptica, tras la identificación de Atiyah del espacio de moduli de fibrados vectoriales de rango n sobre X con el n-producto simétrico $\mathrm{Sym}^n(X)$ de X. Lazlo, Friedman, Morgan y Witten en [35] y [36] construyeron un morfismo biyectivo de $(X \otimes_{\mathbb{Z}} \Lambda_T)/W$ al moduli de G-fibrados $M(G)$ sobre X, donde T es un subgrupo de Cartan de un grupo de Lie reductivo complejo G, Λ_T es la malla de caracteres y $W = N_G(T)/T$ es el grupo Weyl asociado. Como $M(G)$ es normal, este morfismo es un isomorfismo según el teorema principal de Zariski. Thaddeus describió en [81] la normalización $\overline{\mathcal{M}}(G)$ del espacio de moduli de fibrados de Higgs $\mathcal{M}(G)$ como

$$\overline{\mathcal{M}}(G) \cong (T^*X \otimes_{\mathbb{Z}} \Lambda_T)/W .$$

Franco, Garcia-Prada y Newstead identificaron en [33] y [34] el espacio de moduli reducido de G-fibrados de Higgs $\mathcal{M}(G)_{\mathrm{red}}$ con $(T^*X \otimes_{\mathbb{Z}} \Lambda_T)/W$. El espacio de moduli reducido de representaciones del grupo fundamental es isomorfo a $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T/W$ y la correspondencia de Hodge no abeliana es inducida por el difeomorfismo $T^*\widehat{X}$ y $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ dado por la correspondencia de Hodge.

Si X es una superficie de Riemann con algunos puntos, las representaciones en un grupo G con holonomía prescrita alrededor de estos puntos están en correspondencia con fibrados parabólicos poliestables. La correspondencia no abeliana de Hodge fue estudiada primero por Metha y Seshadri en [59] para $G = \mathrm{U}(n)$. Bhosle y Ramanathan en [9], Seshadri y Balaji en [6] y Teleman y Woodward en [83] extendieron la correspondencia para grupos de Lie compactos, considerando holonomías racionales alrededor de los puntos. Cuando G no es compacto, la noción de fibrados de Higgs parabólicos fue introducida por Simpson [76], en el caso $G = \mathrm{GL}(n, \mathbb{C})$. Él construyó en [77] el espacio de moduli de sistemas locales parabólicos filtrados. Si consideramos holonomías racionales, el espacio de moduli de representaciones del grupo fundamental está en correspondencia con los torsores parahóricos, introducidos por [66]. Balaji y Seshadri construyeron en [6] su espacio de moduli y probaron que hay una biyección entre el espacio de moduli de torsores parahóricos y el espacio de moduli de fibrados Γ-equivariantes sobre un cubrimiento cíclico Y de X ramificado sobre cada punto prescrito de X, siendo $Y \to X$ un cubrimiento de Galois de grupo Γ. Biquard, Garcia-Prada y Mundet i Riera introdujeron en [22] **G-fibrados de Higgs parabólicos** para un grupo de Lie G reductivo real. Ellos probaron una correspondencia de Hodge no abeliana con holonomia arbitraria alrededor de los puntos y con la reducción teniendo un comportamiento asintótico prescrito cerca de los puntos. La existencia del espacio de moduli de G-fibrados de Higgs parabólicos se sigue de la construcción general de Schmitt en [73].

Entre los años 2010-2012, se desarrolló la teoría de fibrados vectoriales reales sobre superficies de Klein. Schaffhauser en [71] y [72] y Schaffhauser y Liu en [56] describieron explícitamente los espacios de moduli de estos fibrados. Biswas, Huisman y Hurtubise en [17] introdujeron los G-**fibrados pseudo-reales** sobre variedades Kähler reales, donde G es un grupo de Lie complejo reductivo. Ellos definieron la estabilidad de Behrend de estos objetos y establecieron una correspondencia de Hodge no abeliana entre este espacio de moduli y el espacio de moduli de las representaciones del grupo fundamental orbifold de la variedad de Kähler real. Biswas, Garcia-Prada y Hurtubise extendieron este estudio en [15] y [16] para G-fibrados de Higgs pseudo reales sobre una superficie Klein. Biswas y Garcia-Prada describieron en [14] cómo el espacio de moduli de los fibrados G-fibrados de Higgs pseudo-reales sobre X aparece naturalmente como el conjunto de puntos fijos de involuciones antiholomorfas en el espacio de moduli de G-fibrados de Higgs, dependiendo de las estructuras reales de X y de G. En lenguaje de branas, el espacio de moduli de G-fibrados de Higgs pseudo-reales es una (A, B, A)-branas. Las (A, A, B)-branas fueron estudiadas por Schaposnik y Baraglia en [7] y Biswas y Garcia-Prada en [14].

0.2.2 Resultados principales

Esta tesis trata sobre el estudio de algunos fibrados aumentados equipados con una estructura real. Más específicamente, estudiamos: pares reales de Higgs, G-fibrados de Higgs reales y G-fibrados de Higgs parabólicos reales, donde G es forma real de un grupo $G^{\mathbb{C}}$ semisimple. Estos fibrados generalizan la noción de fibrados pseudo-reales de Higgs dada en [17]. Probamos que estos fibrados aparecen de manera natural como puntos fijos de involuciones en los espacios de moduli de pares de Higgs, de G-fibrados de Higgs y de G-fibrados de Higgs parabólicos, respectivamente. También probamos correspondencias de Hodge no abelianas para G-fibrados de Higgs reales y para G-fibrados de Higgs parabólicos reales.

Para lograr estos objetivos, primero introducimos los L-**pares de Higgs** $(c, \sigma_L, \pm)$-**reales (cuaterniónicos)** (E, φ, σ_E) sobre una superficie de Klein compacta (X, σ_X), que dependen de: una estructura real σ_G en un grupo complejo de Lie reductivo G, una estructura real (cuaterniónica) σ_L en un fibrado de línea L, una estructura real (cuaterniónica) $\sigma_{\mathbb{V}}$ en $\mathbb{V}$, un elemento c en el centro Z de G invariante por σ_G de orden dos y un parámetro $\pm$. Definimos la estabilidad de Behrend, considerando solo reducciones tales que el fibrado asociado adjunto del fibrado reducido es preservado por $\sigma_{\mathrm{Ad}(E)}$, la involución antiholomorfa inducida por σ_E y σ_G.

Una **reducción Hermite-Einstein-Higgs σ_E-compatible** del grupo de estructura de un L-par de Higgs (E, φ, σ_E) de G a H es una reducción h del grupo de estructura de E de G a un subgrupo compacto maximal H de G, de modo que el fibrado reducido se preserva mediante σ_E y satisface la ecuación 0.2.5.

Teorema 2.5.1 [correspondencia de Hitchin-Kobayashi] Un par de Higgs $(\sigma_X, \sigma_G, c, \sigma_L, \pm)$-real (cuaterniónico) (E, φ, σ_E) es α-polystable si y sólo si existe una reducción Hermite-Einstein-Higgs σ_E-compatible.

Los **G-fibrados de Higgs $(\sigma_X, \sigma_G, c, \pm)$-reales** son casos particulares de pares de Higgs, donde G es una forma real semisimple de un grupo de Lie conexo reductivo $G^{\mathbb{C}}$ equipada con una conjugación μ de $G^{\mathbb{C}}$ tal que $(G^{\mathbb{C}})^{\mu} = G$ y con una conjugación τ de $G^{\mathbb{C}}$ tal que $(G^{\mathbb{C}})^{\tau} = H^{\mathbb{C}}$ es un subgrupo compacto maximal de $G^{\mathbb{C}}$. En este caso, σ_G es una conjugación (μ, τ)-compatible (Subsección 1.3.3) y c es un elemento de orden dos, invariante por σ_G que vive en la intersección de los centros de $H^{\mathbb{C}}$ y $G^{\mathbb{C}}$. Denotamos por $\mathcal{M}(\sigma_X, \sigma_G, c, \pm)$ al espacio de G-fibrados de Higgs reales y por $\mathcal{R}(\sigma_X, \sigma_G, c, \pm)$ al espacio de moduli de representaciones $(\sigma_X, \sigma_G, c, \pm)$-compatibles del grupo fundamental orbifold, donde σ_G es una conjugación (μ, τ)-compatible de una forma real (Subsección 1.3.3).

Teorema 3.4.6 [correspondencia de Hodge no abeliana] Hay una correspondencia bijectiva entre $\mathcal{M}(\sigma_X, \sigma_G, c, \pm)$ y $\mathcal{R}(\sigma_X, \sigma_G, c, \pm)$.

Consideremos las siguientes involuciones en el espacio de moduli de G-fibrados de Higgs $\mathcal{M}$

$$\begin{array}{rccc} i_{\mathcal{M}}(\sigma_X, \sigma_G)^{\pm}: & \mathcal{M} & \longrightarrow & \mathcal{M} \\ & (E, \varphi) & \longmapsto & (\sigma_X^* \sigma_G E, \pm \sigma_X^* \sigma_G \varphi)\,. \end{array} \tag{0.2.6}$$

Hay una inmersión de $\mathcal{M}(\sigma_X, \sigma_G, c, \pm)$ en $\mathcal{M}$. Denotamos a su imagen por $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \pm)$.

Teorema 3.6.1 Los puntos fijos de $i_{\mathcal{M}(\alpha)}(\sigma_X, \sigma_G)^{\pm}$ y el espacio de moduli de G-fibrados de Higgs reales se relacionan así:

(1)

$$\mathcal{M}^{i_{\mathcal{M}}(\sigma_X,\sigma_G)^{\pm}} \supseteq \bigcup_{c\in Z(H^{\mathbb{C}})_2^{\sigma_G}\cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X,\sigma_G,c,\pm)\,.$$

(2) For $g(X)\geq 2$ y si restringimos al locus diferenciable sm, entonces

$$\mathcal{M}_{\mathrm{sm}}^{i_{\mathcal{M}}(\sigma_X,\sigma_G)^{\pm}} \cong \bigcup_{c\in Z(H^{\mathbb{C}})_2^{\sigma_G}\cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X,\sigma_G,c,\pm)_{\mathrm{sm}}\,.$$

Gracias al Teorema 3.4.6, probamos un resultado similar en el caso del espacio de moduli de representaciones del grupo fundamental orbifold en G.

Sea $\mathcal{M}(Z)$ el espacio de moduli de Z-fibrados principales topológicamente triviales bundles sobre una curva elíptica, donde Z es el centro de G. $\mathcal{M}(Z)$ tiene estructura de grupo de Lie complejo. Sea $\mathcal{F}=(F,\phi)\in\mathcal{M}(Z)$ tal que $\mathcal{F}=\mathcal{F}^{-1}$. Definimos la siguiente involución en el espacio de moduli de G-fibrados de Higgs, donde G es un grupo de Lie complejo reductivo

$$\begin{array}{rccc} i_{\mathcal{M}}(\alpha_\epsilon,\sigma_\epsilon,\pm,\mathcal{F}): & \mathcal{M}(G) & \longrightarrow & \mathcal{M}(G) \\ & (E,\varphi) & \longmapsto & \mathcal{F}\otimes\iota_{\mathcal{M}}(\alpha_\epsilon,\sigma_\epsilon,\pm)(E,\varphi)\,, \end{array}$$

donde α_ϵ, σ_ϵ con involuciones holomorfas ó antiholomorfas en X y G respectivamente y el parametro ϵ indica cuando es holomorfa($\epsilon=+$) y cuando es anti-holomorfa ($\epsilon=-$). Consideremos la involución

$$\begin{array}{rccc} \dot{I}(\alpha_\epsilon,\sigma_\epsilon,\pm,\mathcal{F}): & T^*\widehat{X}\otimes_{\mathbb{Z}}\Lambda_T & \longrightarrow & T^*\widehat{X}\otimes_{\mathbb{Z}}\Lambda_T \\ & \sum(L_i,\psi_i)\otimes\lambda_i & \longmapsto & [\mathcal{F}]+\sum i_{\mathcal{M}}(\alpha_\epsilon,\alpha_\epsilon,\pm)(L_i,\psi_i)\otimes\sigma_\epsilon(\lambda_i)\,. \end{array}$$

En el Corolario 4.5.2, describimos explícitamente el conjunto de puntos fijos de las involuciones:

$$\mathcal{M}(G)^{\iota_{\mathcal{M}}(t_y\circ\alpha_\epsilon,\sigma_\epsilon,\pm,\mathcal{F})} = \bigcup_{\overline{\omega}\in W/_{\sigma_\epsilon}W} \left(T^*\widehat{X}\otimes_{\mathbb{Z}}\Lambda_T\right)^{\omega\left(\dot{I}(\alpha_\epsilon,\sigma_\epsilon,\pm,\mathcal{F})\right)}/N_W(T^{\omega\sigma_\epsilon})\,.$$

Probamos un resultado similar para el espacio de moduli de representaciones del grupo fundamental de X. Usando este resultado describimos explícitamente el espacio de moduli de fibrados de Higgs $(\sigma_X,\sigma_G,c,\pm)$-reales.

Sea σ_G una conjugación (μ,τ)-compatible de una forma real G de $G^{\mathbb{C}}$. Sea $S=\{x_1,\ldots,x_r\}$ un conjunto finito de puntos de una superficie de Klein compacta

(X, σ_X), tal que $\sigma_X(S) = S$. Sea $\mathcal{A}$ una alcova invariante por σ_G. Sea α un conjunto de pesos en $\sqrt{-1}\mathcal{A}$ que sea (S, σ_G)-compatible (Definición 5.2.2). Un **G-fibrado de Higgs parabolico $(\sigma_X, \sigma_G, c, \alpha, \pm)$-real** es un G-fibrado de Higgs $(\sigma_X, \sigma_G, c, \pm)$-real tal que las structuras parabolicas de peso α sobre un conjunto de puntos S son $(S, \sigma_{\mathrm{Ad}(E)})$-compatible and el campo de Higgs es invariant bajo la involución $P\mathrm{Ad}(f)$ inducida por σ_G. Denote por $\mathcal{M}(\sigma_X, \sigma_G, \alpha, \pm, \mathcal{L})$ al espacio de moduli de clases de equivalencia meromórfica de G-fibrados de Higgs parabólicos $(\sigma_X, \sigma_G, c, \alpha, \pm)$-reales polistables tales que la proyección $\mathrm{Gr}\,\mathrm{Res}_{x_i}\,\varphi$ del residuo de φ en x_i es $\mathcal{L}_i$ y $\mathcal{L} := (\mathcal{L}_1, \ldots, \mathcal{L}_r)$ es $(S, \mathrm{Ad}(f))$-compatible. Un **sistema local parabólico filtrado $(\sigma_X, \sigma_G, c, \beta, \pm)$-compatible** es un triple (F, S, σ_F) tal que $F_{X\setminus S}$ es un sistema local equipado con una estructura $(\sigma_X, \sigma_G, c, \pm)$-compatible (Definición 3.3.2) y sobre cada punto prescrito x_i y cada rayo ρ_i yendo a x_i hay un par (P_i, χ_i) , donde P_i es un subgrupo parabólico de $F(G)|_{\rho_i}$ isomorfo a P_{β_i}, P_i es invariante bajo la trasformación monodromía alrededor de x_i, σ_F y σ_G inducen una involución $\sigma_{F(G)}$ en $F(G)$ tal que $(P_1, \ldots, P_r) \in \prod_{i=1}^r F(G)|_{\rho_i}$ es $(S, \sigma_{F(G)})$-compatible y χ_i es un carácter estrictamente antidominante de P_i.

Denotamos por $\mathcal{S}(\sigma_X, \sigma_G, c, \beta, \pm, \mathcal{C})$ el espacio de moduli de sistemas locales filtrados de peso β con monodromias $\mathcal{C} = (\mathcal{C}_1, \cdots, \mathcal{C}_r)$, siendo $\mathcal{C}$ un elemento (S, σ_G)-compatible. Denotamos por $\mathcal{R}(\sigma_X, \sigma_G, c, \pm, \mathcal{C})$ el espacio de moduli de representaciones reductivas del grupo orbifold fundamental de $X\setminus S$ en G, tales que la restricción a $\pi_1(X\setminus S)$ tiene como imágenes de los lazos alrededor de los puntos prescritos son clases de conjugación $\mathcal{C}$, siendo $\mathcal{C} \in \prod_{i=1}^r \mathrm{Conj}(G)$ un elemento (S, σ_G)-compatible.

Teorema 5.7.3 [correspondencia de Hodge no-abeliana] Si los pesos y las monodromías están relacionadas por la Tabla 5.1, entonces:

(1) $\mathcal{M}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L})$ es homeomorfo a $\mathcal{S}(\sigma_X, \sigma_G, c, \beta, \pm, \mathcal{C})$.

(2) $\mathcal{S}(\sigma_X, \sigma_G, c, 0, \pm, \mathcal{C})$ es homeomorfo a $\mathcal{R}(\sigma_X, \sigma_G, c, \pm, \mathcal{C})$.

La involución $i_{\mathcal{M}}(\sigma_X, \sigma_G)^{\pm}$ definida en (0.2.6) se puede extender al espacio de moduli de fibrados parabolicos $\mathcal{M}(\alpha, \mathcal{L})$. Denotemos dicha extensión por $i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X, \sigma_G)^{\pm}$.
Theorem 5.8.1 Los puntos fijos de la involution $i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X, \sigma_G)^{\pm}$ y el espacio de

moduli de G-fibrados de Higgs parabólicos $(\sigma_X, \sigma_G, c, \alpha, \pm)$-reales de clase $\mathcal{L}$ están relacionados por:

(1)
$$\mathcal{M}(\alpha, \mathcal{L})^{i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X,\sigma_G)^{\pm}} \supseteq \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L}) \,.$$

(2) Para $g(X) \geq 2$ y si restringimos al locus diferenciable sm, entonces
$$\mathcal{M}(\alpha, \mathcal{L})_{\mathrm{sm}}^{i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X,\sigma_G)^{\pm}} \cong \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L})_{\mathrm{sm}} \,.$$

Gracias al Teorema 5.7.3 probamos resultados similares para el espacio de moduli de sistemas locales filtrados parabólicos y para el espacio de moduli de representaciones del grupo fundamental en G.

0.2.3 Contenidos

El capítulo 1 está dedicado a introducir estructuras reales para los objetos principales estudiados en esta tesis. En la Sección 1.1, recordamos la definición y clasificación topológica de las superficies Klein compactas. En la Subsección 1.1.1, recordamos la clasificación de curvas elípticas reales y describimos acciones de estructuras reales en $\pi_1(X)$. En la Subsección 1.1.2, definimos formas Kähler reales y probamos que si una superficie de Klein (X, σ_X) está equipada con una forma Kähler, entonces admite una forma Kähler σ_X-real. En la Sección 1.2 definimos estructuras reales, cuaterniónicas y reales unitarias en espacios vectoriales complejos.

La Sección 1.3 está dividida en tres partes, en la primera, definimos estructuras reales sobre álgebras de Lie y su relación con automorfismos de orden dos, establecida por Cartan. En la segunda parte, estudiamos estructuras reales en grupos de Lie y su relación con automorfismos holomorfos del grupo (Teorema de Cartan). Como consecuencia, dada una estructura real σ_G en un grupo G, existe un subgrupo compacto maximal de G invariante bajo σ_G. En la tercera parte, definimos conjugaciones (μ, τ)-compatibles de $G^{\mathbb{C}}$ y estudiamos sus propiedades básicas.

Definimos acciones reales de grupos en variedades, ambos equipados con estructuras reales, en la Sección 1.4. Las representaciones lineales reales son un ejemplo particular, que después generalizamos a representaciones cuaterniónicas y real unitarias.

En la Sección 1.5, definimos estructuras unitarias reales, cuaterniónicas y reales en un fibrado vectorial E. En la Subsección 1.5.1, reformulamos estas definiciones en términos del operador de Dolbeaut del fibrado C^∞ subyacente $\mathbb{E}$. Las métricas hermiticas compatibles en $\mathbb{E}$ se definen en la Subsección 1.5.2. En la Subsección 1.5.3, definimos conexiones compatibles en $\mathbb{E}$ y probamos que si fijamos una métrica hermítica compatible en $\mathbb{E}$, entonces hay una correspondencia biyectiva entre los operadores Dolbeaut reales, cuaterniónicos y reales unitarios y las conexiones compatibles en $\mathbb{E}$. En la Subsección 1.5.4, recordamos la clasificación topológica dada en [17, p.8] de los fibrados vectoriales reales, cuaterniónicos y real unitarios sobre una superficie Klein.
La Sección 1.6 trata sobre G-bundles reales, donde G es un grupo de Lie reductivo complejo. En la Subsección 1.6.1, definimos G-bundles reales en términos de la estructura holomorfa del fibrado C^∞ subyacente $\mathbb{E}$. Las reducciones y conexiones compatibles se definen en la Subsección 1.6.2 y probamos que si fijamos una reducción compatible en $\mathbb{E}$, hay una correspondencia biyectiva entre estructuras holomorfas reales y las conexiones compatibles. En la Subsección 1.6.3, recordamos la clasificación topológica de G-fibrados pseudo-reales dada en [17, p.13]. Estudiamos estructuras reales en fibrados asociados y damos algunos ejemplos en la Sección 1.7.

En el Capítulo 2, generalizamos el estudio de pares de Higgs realizado en [39], [61] y [25] para incorporar el caso en el que todos los objetos tienen una estructura real. Primero, definimos la estructura real en los pares de Higgs y damos algunos ejemplos en la Sección 2.1. En la Sección 2.2, recordamos la ecuación de Hermite-Einstein-Higgs e incluimos la estabilidad de Behrend para pares de Higgs reales (cuaterniónicos) en la Sección 2.3. En la Sección 2.4, recordamos de [62, Sección 2.3.3], que los L-pares de Higgs se pueden considerar como una variedad simpléctica infinito dimensional, la acción del grupo gauge es hamiltoniana y la aplicación momento es la ecuación de Hermite-Einstein-Higgs y el grado de un par de L-Higgs se puede estimar como un peso máximal.
Enunciamos la correspondencia de Hitchin-Kobayashi para pares de Higgs reales (cuaterniónicos) en la Sección 2.5. En la Sección 2.6, definimos el espacio de automorfismos infinitesimales semisimples de un par de Higgs real (cuaterniónico) y

probamos que el par de Higgs real (cuaterniónico) es estable si y sólo si los automorfismos infinitesimales son centrales. En la Sección 2.7, demostramos que un par de Higgs real (cuaterniónico) poliestable admite una reducción Jordan-Hölder real y como resultado da un par de Higgs real (cuaterniónico) estable.

En la Sección 2.8 demostramos que la estabilidad en pares de Higgs simples reales (cuaterniónicos) implica la existencia de soluciones de la ecuación de Hermite-Einstein-Higgs, siguiendo las ideas in [61]. Más concretamente, primero definimos pares de Higgs reales (cuaterniónicos) simplificados en la Subsección 2.8.1. Los pares de Higgs reales (cuaterniónicos) se pueden considerar como pares de Higgs simplificados (cuaterniónicos) reales. Definimos estabilidad de pares de Higgs reales (cuaterniónicos) simplificados. En la Subsección 2.8.2, adaptamos el punto de vista simpléctico de la Sección 2.4 para pares de Higgs simplificados. En la Subsección 2.8.3, planteamos nuestro problema en términos de encontrar una métrica en la que se obtenga la integral del mapa del momento un mínimo. En la Subsección 2.8.4, replicamos la prueba en [61].

En la Sección 2.9, probamos que la existencia de soluciones de la ecuación Hermite-Einstein-Higgs implica la polystabilidad del par de Higgs real (cuaterniónico) y en la Sección 2.10 identificamos pares de Higgs reales como puntos fijos de involuciones definidas en pares de Higgs.

Probamos una correspondencia de Hodge no abeliana para G-fibrados de Higgs reales, donde G es un grupo Lie reductivo real, en el capítulo 3. Comenzamos definiendo G-fibrados de Higgs reales como un caso particular de pares de Higgs reales en la Sección 3.1. Por el Teorema 2.5.1, hay una correspondencia biyectiva entre el espacio de móduli de G-fibrados de Higgs reales y el espacio de moduli $\mathcal{A}_0$ de triples que consisten en conexiones compatibles, secciones de Higgs y estructuras reales en el fibrado C^∞ subyacente satisfaciendo las ecuaciones de Hitchin. Esto se hace en la sección 3.2. En la Sección 3.3 probamos una correspondencia biyectiva entre el espacio de moduli de los sistemas locales compatibles y y el espacio de moduli de las representaciones del orbifold grupo fundamental. La correspondencia de Donaldson-Corlette para sistemas locales compatibles $\mathcal{S}$ se demuestra en la Sección 3.4. La correspondencia establece una bijection entre $\mathcal{S}$ y $\mathcal{A}_0$. Este resultado junto con los teoremas de la Sección 3.2 y la Sección 3.3 demuestran la correspondencia de Hodge no abeliana. En la Sección 3.5 definimos involuciones en el espacio de moduli de G-fibrados de Higgs, en el espacio de moduli de sistemas locales y en el espacio de moduli de representaciones del grupo fundamental. Gracias a la correspondencia de Hodge no abeliana para G- fibrados de Higgs reales y a la Sección 2.10, obtenemos una descripción del lugar de puntos fijos de las involuciones en la Sección 3.6.

El Capítulo 4 está dedicado al estudio de G-fibrados de Higgs reales sobre una curva elíptica X. Primero, revisamos en la Sección 4.1 la descripción del espacios de moduli de G-fibrados de Higgs sobre X y el espacio de moduli de representaciones de $\pi_1(X)$ en G. En la Sección 4.2, definimos las involuciones $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ y $J(\alpha_\epsilon, \sigma_\epsilon, \pm)$ sobre estos espacios de moduli, respectivamente. En la Sección 4.3, describimos explícitamente estas involuciones en el caso de fibrados de línea de Higgs. Usando esto, obtenemos una descripción explícita de las involuciones para el caso general en la Sección 4.4. Describimos explícitamente sus lugares de puntos fijos en la Sección 4.5 y el espacio de moduli G-fibrados de Higgs reales sobre X, en la Sección 4.6.

En el Capítulo 5 probamos una correspondencia de Hodge no abeliana para G-fibrados de Higgs parabólicos reales, donde G es un grupo de Lie reductivo real e identificamos estos espacios de moduli como puntos fijos de involuciones. En la Sección 5.1 hacemos un recordatorio de la teoria de G-fibrados de Higgs parabólicos para G real reductive Lie group, siguiendo [22] como referencia. Primero, en la Subsección 5.1.1, revisamos las definiciones del espacio de moduli de G-fibrados de Higgs parabólicos, en la Subsección 5.1.2 del espacio de moduli de sistemas parabólicos locales, en la Subsección 5.1.3 del espacio de moduli de las representaciones de $\pi_1(X \backslash S)$. Estos espacios de moduli dependen de pesos parabólicos, residuos, monodromías y clases de conjugación de G. Concluimos este resumen en la subsección 5.1.4, donde recordamos que estos espacios de moduli están en correspondencia uno a uno eligiendo clases adecuadas. Fijamos una conjugación μ de $G^{\mathbb{C}}$ tal que $(G^{\mathbb{C}})^\mu = G$ y una conjugación τ de $G^{\mathbb{C}}$ tal que $(G^{\mathbb{C}})^\tau = H^{\mathbb{C}}$ es un subgrupo compacto maximal de $G^{\mathbb{C}}$. Definimos en la Sección 5.2 involuciones en estos espacios de moduli dependiendo de una conjugación σ_G de $G^{\mathbb{C}}$ que es (μ, τ)-compatible y una estructura real σ_X en X. Presentamos la definición de fibrados G parabólicos reales en la Sección 5.3 y definimos la estabilidad de Behrend en la Sección de 5.4. Probamos la correspondencia de Hitchin-Kobayashi (Theorem 5.5.5) para G-fibrados de Higgs parabolicos reales en la Sección 5.5. En la sección 5.6, definimos sistemas locales filtrados parabólicos reales y también las nociones de Behrend de z-estabilidad para estos objetos. La correspondencia de Hitchin-Simpson para los sistemas locales filtrados parabólico reales (Teorema 5.7.1) se prueban en la Sección 5.7. Gracias a la correspondencia de Hitchin-Kobayashi y la correspondencia de Hitchin-Simpson (Theorem 5.7.1), hay inmersiones de estos espacios de moduli de objetos reales en los espacios de moduli de G-fibrados de Higgs parabólicos y en el espacio de moduli de sistemas locales parabólicos. Finalmente, demostramos en la Sección 5.8 que los puntos fijos de las involuciones son, precisamente, las imágenes de estas inmersiones.

El Apéndice A está dividido en dos partes. En la primera parte, Sección A.1, siguiendo [44, 50, 53], calculamos la posición relativa de dos símplices con respecto a dos buildings the Tits esféricos equivalentes: el asociado a G y el asociado a la límite en infinito $\partial_\infty(H\backslash G)$ del grupo de cocientes $H\backslash G$. En la segunda parte, Sección A.2, damos dos definiciones de grado parabólico. El grado parabólico de Biquard, Garcia-Prada y Mundet-i-Riera (véase [22]) es la suma del grado habitual y el grado relativo de dos simplices en $\partial_\infty(H\backslash G)$. El grado parabólico de Teleman y Woodward (véase [83]) es la suma del grado habitual y el grado relativo de dos símpleces de la construcción esférica de Tits determinada por G. Probamos que estas dos definiciones son equivalentes.

Chapter 1

Real structures

1.1 Real structures on Riemann surfaces

Definition 1.1.1 A **real structure** σ_X on a Riemman surface X is an antiholomorphic involution of X. The pair (X, σ_X) will be referred as a **Klein surface**. A **morphism of Klein surfaces** $f : (X, \sigma_X) \to (X', \sigma_{X'})$ is a morphism of Riemann surfaces $f : X \to X'$, such that $f \circ \sigma_X = \sigma_{X'} \circ f$.

Remark 1.1.1 Alling and Greanleaf proved in [2, p. 46] that any compact surface can be equipped with a dianalytic atlas (atlas whose charts are dianalytic functions, i.e. the restriction of the function to every connected component is holomorphic or antiholomorhic). In [2, Chapter 6] the authors proved that the category of Klein surfaces is equivalent to the category of dianalytic manifolds of complex dimension one. A Klein surface (X, σ_X) corresponds to the dianalytic manifold X/σ_X. The boundary $\partial(X/\sigma_X)$ of X/σ_X is equal to the set of fixed points X^{σ_X}.

Definition 1.1.2 The **topological parameters** of a Klein surface (X, σ_X) is a triple (g, k, a), where g is the genus of X, k is the number of connected components of X^{σ_X} and a is the index of orientability of X/σ_X, defined as follows

$$a = \begin{cases} 0 & \text{if } X/\sigma_X \text{ is orientable} \\ 1 & \text{if } X/\sigma_X \text{ is not orientable.} \end{cases}$$

This last parameter a can also be defined, equivalently, in the following way.

Proposition 1.1.1 Let (X, σ_X) a Klein surface, with X being connected. The topological parameter a satisfies

$$a = \begin{cases} 0 & \text{if } X - X^{\sigma_X} \text{ has two connected components} \\ 1 & \text{if } X - X^{\sigma_X} \text{ has one connected component.} \end{cases}$$

Proof: We use two properties of the orientability of a manifold M with boundary $\partial(M)$. First, M is orientable if and only if $M - \partial(M)$ is orientable. Second, M is orientable if and only if the double cover of orientations of M has two connected components. We apply the first result to $M = X/\sigma_X$ and the second to $M = X/\sigma_X - \partial(X/\sigma_X)$. $\square$

Proposition 1.1.2 ([70] Proposition 1.3) Let (X, σ_X) a compact Klein surface of topological type (g, k, a). The fixed point set X^{σ_X} is a union of k conected components called ovals and each one is diffeomorphic to a circle.

Proof: Each connected component of X^{σ_X} is compact and belongs to ∂X (see Remark 1.1.1), therefore the charts of the dianalytic associated manifold take values on $\mathbb{R}$, hence they have dimension one. $\square$

Recall that the **Euler characteristic** $\xi(S)$ of any compact surface S is given by the sum of the number of vertices and faces minus the number of edges of any triangulation.

Proposition 1.1.3 Let (X, σ_X) a compact Klein surface, then $\xi(X) = 2\xi(X/\sigma_X)$.

Proof: Any triangulation of X/σ_X, lifts to a triangulation of X in the following way: each face of X/σ_X correspond to two faces of X; each edge or vertice out of the boundary of X/σ_X correspond to two edges or vertices of X and finally, the number of edges or vertices in the boundary of X/σ_X is equal to the number of edges or vertices in X, because by Proposition 1.1.2, each component of the boundary of X/σ_X is a circle.

$\square$

Weichold proved in [86] that two compact Klein surfaces (X, σ_X) and $(X', \sigma_{X'})$ are homeomorphic if X/σ_X and $X'/\sigma_{X'}$ are homeomorphic as dianalytic manifolds. Moreover, he gave a topological classification of Klein surfaces.

Theorem 1.1.3 (Weichold) Two compact Klein surfaces are homeomorphic if and only if they have the same topological parameters.

The topological parameters of a Klein suface are not arbitrary, there are some constrictions, given by the following.

Theorem 1.1.4 A compact Klein suface (X, σ_X) exists if and only if their topological parameters (g, k, a) satisfy the following conditions:

(1) $0 \leq k \leq g+1$,

(2) if $k = g+1$, then $a = 0$,

(3) if $a = 1$, then $0 \leq k \leq g$,

(4) if $a = 0$, then $k \equiv g+1 \mod 2$,

(5) if $k = 0$, then $a = 1$.

Proof:

Let (X, σ_X) be a compact Klein suface of topological parameters (g, k, a). We follow [75] to prove the conditions above. We denote by S^2_{p+l}, T_1 and $\mathbb{R}P^2_1$, the sphere with $p + l$ open disjoint disks, the torus minus a disk and the real projective plane minus a disk, respectively. The Euler characteristics of these surfaces are given by

$$\xi(\mathbb{R}P^2_1) = 0 \ , \ \xi(T_1) = -1 \text{ and } \xi(S^2_{p+l}) = 2 - p - l.$$

Any compact surface $\Sigma_{p,l}$ can be built by taking S^2_{p+l} and gluing to it p copies of T_1 and l copies of $\mathbb{R}P^2_1$. The Euler character of $\Sigma_{p,l}$ is

$$\xi(\Sigma_{p,l}) = \xi(S^2_{p+l}) + p \cdot \xi(T_1) + l \cdot \xi(\mathbb{R}P^2_1) = 2 - 2p - l.$$

Any compact surface $\Sigma_{p,l,k}$ with k componected components in the boundary is obtained deleting k open disks with disjoint closures from a surface of type $\Sigma_{p,l}$. Therefore,

$$\xi(\Sigma_{p,l,k}) = 2 - 2p - l - k\,. \tag{1.1.1}$$

The Euler characteristic of the dianalytic manifold X/σ_X is equal to $1-g$, by Proposition 1.1.3. By Equation (1.1.1), one has $\xi(\Sigma_{p,l,k}) = 2-2p-l-k$, then

$$\xi(\Sigma_{p,l,k}) = 2-2p-l-k = 1-g. \tag{1.1.2}$$

By (1.1.2) and the fact that $p, l \geq 0$, we obtain that $1-g \leq 2-k$ and this proves (1). To prove (2), we use that (1.1.2) implies that $l=-2p$, hence $l=2p=0$. Since $p=l=0$, then X/σ_X is orientable. Statement (3) is consequence of (1) and (2). If $a=0$, then X/σ_X is a manifold of the form $\Sigma_{p,0,k}$, so $1-g=2-2p-k$, therefore $k \equiv g+1 \mod 2$ and this proves (4). Finally, (5) is consequence of the definition of a given in Proposition 1.1.1.

Conversely, we follow [70, p.7] to construct Klein sufaces satisfying the different types of restrictions. First case: $a=0$. Let k be a number such that $g+1-k$ is even, we consider a compact Riemann surface Y of genus $\frac{g+1-k}{2}$ with k open disks delete from it. Let $\bar{Y}$ be the Riemann surface obtained from Y by replacing the complex structure of Y with its conjugate structure, i.e. by replacing all local variables z with their complex conjugates $\bar{z}$. Glueing the surfaces Y and $\bar{Y}$ together identifying the boundary points, one obtains a compact Riemann X surface of genus g. The identity mapping $Y \to \bar{Y}$ induces an antiholomorphic involution $\sigma : X \to X$ and we obtain a Klein suface (X, σ) with prescribed (g, k, a).

Second case: $a=1$. In this case, the compact Klein surface is homeomorphic to $\Sigma_{p,l,k}$, with $l \geq 1$. We need to construct a doble covering S and an antiholomorhic involution σ_S such that S/σ_S is homeomorphic to $\Sigma_{p,l,k}$. Let $\Sigma_{p,0,k+l} := \Sigma_{p,l,k} - l \cdot \mathbb{R}P^2_1$. The boundary $\partial(\Sigma_{p,0,k+l})$ is equal to the disjoint union of k circles $A_1, \ldots, A_k$ and l circles $B_1, \ldots, B_l$ coming from $\mathbb{R}P^2_1$. Each B_i is equipped with a fixed point free involution σ_{B_i}. We define

$$S := \Sigma^1_{p,0,k} \cap \Sigma^2_{p,0,k} / \sim ,$$

where $\Sigma^1_{p,0,k+l}$ and $\Sigma^2_{p,0,k+l}$ are two copies of $\Sigma_{p,0,k+l}$ and we identify each point $x \in A^1_i$ with $x \in A^2_i$ and each point $y \in B^1_i$ with $\sigma_{B_i}(y) \in B^2_i$. The map sending $x \in \Sigma^1_{p,0,k+l}$ to $x \in \Sigma^2_{p,0,k+l}$ induces an antiholomorhic involution σ_S on S and S/σ_S is homeomorphic to $\Sigma_{p,l,k}$. □

Taking the different types of triples (g, k, a) satisfying the conditions above we obtain all the non homeomorphic Klein surfaces. For instance, if g is zero or one they are described in the following table.

g	k	a	Notation	Name
0	0	1	$\mathbb{R}P^2$	Real proyective plane
0	1	0	$\overline{\mathtt{D}}$	Closed Unit disc
1	0	1	$\mathtt{K}$	Klein bottle
1	1	1	$\mathtt{M}$	Moebius strip
1	2	0	$\overline{\mathtt{A}}$	closed annulus

1.1.1 Real elliptic curves

Definition 1.1.5 An **elliptic curve** is a pair (X, x_0), where X is a compact Riemann surface of genus 1 and x_0 is a distiguished point on it.

By abuse of notation, we refer to the elliptic curve simply as X. We denote by $\mathcal{H}$ the upper half plane. Every elliptic curve is isomorphic to some $X_\gamma := \mathbb{C}/\langle 1, \gamma\rangle_{\mathbb{Z}}$, where $\langle 1, \gamma\rangle_{\mathbb{Z}}$ is the lattice generated by 1 and $\gamma \in \mathcal{H}$. Two elliptic curves defined by $\gamma, \gamma' \in \mathcal{H}$ are isomorphic if γ, γ' are related by a Möbius transformation. Therefore, we can suppose that γ lies in $\mathcal{H}/\operatorname{PSL}(2, \mathbb{Z})$, that is isomorphic to

$$\Omega = \left\{ \gamma \in \mathcal{H} \ : \ \Re(\gamma) \in \left[-\frac{1}{2}, \frac{1}{2}\right], \ \Im(\gamma) \geq \sqrt{1 - \Re(\gamma)^2} \right\}. \tag{1.1.3}$$

where $\Re(\gamma)$ and $\Im(\gamma)$ denote the real and imaginary parts of a complex number γ, respectively. One of the fundamental properties of an elliptic curve (X, x_0) is that it is equipped with a canonical abelian group structure given by $\widehat{X} := \operatorname{Pic}^0(X)$, the Picard group of degree 0. Indeed, the Abel-Jacobi morphism of degree 1

$$\begin{array}{rrcl} \mathrm{aj}_1 : & X & \longrightarrow & \operatorname{Pic}^1(X) \\ & x & \longmapsto & \mathcal{O}_X(x) \end{array}$$

in the case of elliptic curves is an isomorphism and tensoring by $\mathcal{O}(x_0)^{-1}$ gives the isomorphism

$$p_{x_0} : X \xrightarrow{\cong} \widehat{X}, \tag{1.1.4}$$

which induces a group structure on X, with x_0 as the neutral element. Using this group structure, one can define for each $y \in X$ the **traslation by** y as

$$\begin{array}{rccc} t_y : & X & \longrightarrow & X \\ & x & \longmapsto & x+y\,. \end{array}$$

There is a bijective correspondence between the Jacobian variety $\widehat{X_\gamma}$ and representations of the fundamental group $\pi_1(X_\gamma, x_0)$ in the unit circle S^1. The fundamental group $\pi_1(X_\gamma)$ is given by the homotopy classes of δ_1 and δ_2 generated by the projections of $\tilde{\delta}_1, \tilde{\delta}_2 : [0,1] \to \mathbb{C}$, where

$$\tilde{\delta}_1(z) = z \tag{1.1.5}$$

and

$$\tilde{\delta}_2(z) = \gamma z\,. \tag{1.1.6}$$

Definition 1.1.6 A **real elliptic curve** is a triple (X, x_0, σ_X), where (X, x_0) is an elliptic curve and σ_X is an antiholomorphic involution on X.

Let (X, x_0, σ_X) a real elliptic curve. We denote by $\tilde{\sigma}_X : \mathbb{C} \to \mathbb{C}$ the lift of σ_X to the universal covering. It has the form $\tilde{\sigma}_X(z) = a\bar{z} + b$, where a, b are complex numbers. By imposing the conditions on a, b for $\tilde{\sigma_X}$ to be well defined, Alling and Greenleaf proved in [2, Section 9] that there only exist antiholomorphic involutions on the regions of Ω (up to Möbius transformation) specified in Table 1.1.

Region	Values of γ
A	$\{\Im(\gamma) > 1, \Re(\gamma) = 0\}$
B	$\{\Im(\gamma) = 1, \Re(\gamma) = 0\}$
C	$\{0 < \Re(\gamma) < \frac{1}{2}, \Im(\gamma) = \sqrt{1 - \Re(\gamma)}\}$
D	$\{\Im(\gamma) = \frac{\sqrt{3}}{2}, \Re(\gamma) = \frac{1}{2}\}$
E	$\{\Im(\gamma) > \frac{\sqrt{3}}{2}, \Re(\gamma) = \frac{1}{2}\}$

Table 1.1: X_γ admitting an antiholomorphic involution.

We denote by $\sigma_{-1,a}$ the antiholomorphic involution on X_γ that fixes the identity and lifts to $\mathbb{C} \to \mathbb{C}$, $z \mapsto a\bar{z}$, where $a \in \mathbb{C}^*$. The antiholomorphic involution $t_y \circ \alpha_a$

lifts to $z \mapsto a\bar{z} + b$, where $y \in X$ is the projection of b to X_γ. The Table 1.2 contains any possible antiholomorphic involution of an elliptic curve X_γ; it gives the set possible values of y depending on a and the region on which the real elliptic curve $(X_\gamma, t_y \circ \sigma_a)$ lives.

Region	Involution	Admissible traslations	Topological type
A	$\alpha_{(-1,\pm 1)}$		(2,0)
	$t_y \circ \alpha_{(-1,\pm 1)}$	$y \neq x_0, y \in X^{\alpha_{(-1,\mp 1)}} \cong S^1 \sqcup S^1$	(0,1)
B	$\alpha_{(-1,\pm 1)}$		(2,0)
	$\alpha_{(-1,\pm\sqrt{-1})}$		(1,1)
	$t_y \circ \alpha_{(-1,\pm 1)}$	$y \neq x_0, y \in X^{\alpha_{(-1,\mp 1)}} \cong S^1 \sqcup S^1$	(0,1)
C	$t_y \circ \alpha_{(-1,\pm\gamma)}$	$y \in X^{\alpha_{(-1,\mp\gamma)}} \cong S^1$	(1,1)
D	$t_y \circ \alpha_{(-1,\pm 1)}$	$y \in X^{\alpha_{(-1,\mp 1)}} \cong S^1$	(1,1)
	$t_y \circ \alpha_{(-1,\pm\gamma)}$	$y \in X^{\alpha_{(-1,\mp\gamma)}} \cong S^1$	(1,1)
	$t_y \circ \alpha_{(-1,\pm\gamma^2)}$	$y \in X^{\alpha_{(-1,\mp\gamma^2)}} \cong S^1$	(1,1)
E	$t_y \circ \alpha_{(-1,\pm 1)}$	$y \in X^{\alpha_{(-1,\mp 1)}} \cong S^1$	(1,1)

Table 1.2: Antiholomorphic involutions on X_γ.

In the above cases, the table registers some real structures that are equivalent in the sense that the associated Klein surfaces are isomorphic. Let us see the different isomorphism in each region:

Proposition 1.1.4 (1) In the region A, for any $y \in X^{\alpha_{(-1,\mp 1)}}$ such that $y \neq x_0$,

$$\left(X_\gamma, t_y \circ \alpha_{(-1,1)}\right) \cong \left(X_\gamma, t_{\frac{1}{2}} \circ \alpha_{(-1,1)}\right) ,$$

and

$$\left(X_\gamma, t_y \circ \alpha_{(-1,-1)}\right) \cong \left(X_\gamma, t_{\frac{\gamma}{2}} \circ \alpha_{(-1,1)}\right) .$$

(2) In the region B,

$$\left(X_\gamma, \alpha_{(-1,1)}\right) \cong \left(X_\gamma, \alpha_{(-1,-1)}\right) ,$$

$$\left(X_\gamma, \alpha_{(-1,\mathbf{i})}\right) \cong \left(X_\gamma, \alpha_{(-1,-\mathbf{i})}\right) ,$$

and for every $y \in X^{\alpha_{(-1,\mp 1)}}$ such that $y \neq x_0$,

$$\left(X_\gamma, t_y \circ \alpha_{(-1,1)}\right) \cong \left(X_\gamma, t_{\frac{1}{2}} \circ \alpha_{(-1,1)}\right) .$$

(3) In the regions C, D and E, for all $y \in X^{\alpha_{(-1,\mp a)}}$,

$$\left(X_\gamma, t_y \circ \alpha_{(-1,a)}\right) \cong \left(X_\gamma, \alpha_{(-1,a)}\right) .$$

(4) In the region D,

$$\left(X_\gamma, \alpha_{(-1,\pm 1)}\right) \cong \left(X_\gamma, \alpha_{(-1,\mp\gamma)}\right) \cong \left(X_\gamma, \alpha_{(-1,\pm\gamma^2)}\right) .$$

Recall the generators δ_1, δ_2 of the fundamental goup $\pi_1(X)$ defined in (1.1.5) and (1.1.6). The following Table 1.3 studies the action of $\sigma_{-1,a}$ on $\pi_1(X)$.

Region	Involution	$(\alpha_{(c,a)})_*(\delta_1,\delta_2)$
A, B	$\alpha_{(-1,1)}$	$(\delta_1, -\delta_2)$
	σ_{-1}	$(-\delta_1, \delta_2)$
C, D	$\alpha_{(-1,1)}$	$(\delta_1, -\delta_2)$
	σ_{-1}	$(-\delta_1, \delta_2)$
	σ_γ	(δ_2, δ_1)
	$\sigma_{-\gamma}$	$(-\delta_2, -\delta_1)$
E	$\alpha_{(-1,1)}$	$(\delta_1, -\delta_2 + \delta_1)$
	σ_{-1}	$(-\delta_1, \delta_2 - \delta_1)$

Table 1.3: Action of α_a on $\pi_1(X)$.

For completeness, we give two tables that are the analogous to the previous ones, but in the holomorphic case. These will be used in Chapter 5. To present a consistent notation, we write $\alpha_{(+,1)}$ for the identity Id_X and $\alpha_{(+,-1)}$ for the inversion map $x \longmapsto -x$ of X. When they are simultaneously referred, we shall write $\alpha_{(+,a)}$. Composing $\alpha_{(+,a)}$ with the translations t_y one gets another involution provided $y = -\alpha_{(+,a)}(y)$. For an involution α of X, let $X^\alpha \subset X$ be the fixed point locus and X/α the quotient by α. The first table describes the holomorphic involutions of X_γ and the second studies the action of $\sigma_{1,a}$ on $\pi_1(X)$.

Involution	Admissible translations	X^α	X/α
$\alpha_{(+,1)}$	$-$	X_γ	X_γ
$t_y \circ \alpha_{(+,1)}$	$y \in X[2], y \neq x_0$	$\emptyset$	$X_{(\gamma-\widetilde{y})}$
$\alpha_{(+,-1)}$	$-$	$X[2]$	$\mathbb{P}^1$
$t_y \circ \alpha_{(+,-1)}$	$y \in X$	$\frac{1}{2}y + X[2]$	$\mathbb{P}^1$

Table 1.4: Holomorphic involutions of X_γ.

Involution	$\big((\alpha_{\alpha_\epsilon})_*\delta_1, (\alpha_{(\epsilon,a)})_*\delta_2\big)$
$\alpha_{(+,1)}$	(δ_1, δ_2)
$\alpha_{(+,-1)}$	$(-\delta_1, -\delta_2)$

Table 1.5: Action of $\alpha_{(+,a)}$ on $\pi_1(X_\gamma)$.

1.1.2 Real Kähler forms

Definition 1.1.7 Let (X, σ_X) be a Klein surface. A Kähler form ω on X is σ_X-**real** if $\sigma_X^*\omega = -\omega$.

Proposition 1.1.5 ([15], p. 4) Let (X, σ_X) be a Klein surface equipped with a Kähler form ω. There always exists a σ_X-real Kähler form on (X, σ_X).

Proof: Let J be a complex structure on X. We define the inner product

$$g_\omega(u, v) := \omega(u, Jv)\,,$$

for all $u, v \in T^{\mathbb{R}}X$, where $T^{\mathbb{R}}X$ means the tanget space of the underlying real manifold. A Kähler form ω' is σ_X-real if and only if

$$\sigma_X^* g_{\omega'} = g_{\omega'}\,. \tag{1.1.7}$$

We define $g' = g_\omega + \sigma_X^* g_\omega = g_{\omega - \sigma_X^*\omega}$. The inner product g' satisfies (1.1.7), therefore $\omega' := \omega - \sigma_X^*\omega$ is a σ_X-real Kähler form. □

1.2 Real structures on vector spaces

Definition 1.2.1 Let $\mathbb{V}$ be a complex vector space of complex dimension n. A **real structure** $\sigma_{\mathbb{V}}$ on $\mathbb{V}$ is an antilinear map $\sigma_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}$ such that $\sigma_{\mathbb{V}}^2 = \mathrm{Id}_{\mathbb{V}}$.

Example 1.2.1 Let $\mathbb{V} = \mathbb{C}^n$ and $\sigma_{\mathbb{V}}(z_1, \ldots, z_n) := (\overline{z_1}, \ldots, \overline{z_n})$, for any vector $(z_1, \ldots, z_n) \in \mathbb{V}$, where $\overline{z_i}$ denotes the conjugation of the complex number z_i, for all $1 \leq i \leq n$. The involution $\sigma_{\mathbb{V}}$ is a real structure on $\mathbb{C}^n$.

Definition 1.2.2 Let $n = 2m$ be an even number and $\mathbb{V}$ a complex vector space of dimension n. We denote by I_m the identity matrix of rank m. A **quaternionic structure** on $\mathbb{V}$ is an antilinear map $\theta_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}$ with respect to

$$\mathrm{I} = \begin{pmatrix} \sqrt{-1}\,\mathrm{I}_m & 0 \\ 0 & -\sqrt{-1}\,\mathrm{I}_m \end{pmatrix} \tag{1.2.8}$$

such that $\sigma_{\mathbb{V}}^2 = -\mathrm{Id}_{\mathbb{V}}$.

Remark 1.2.1 If $\mathbb{V}$ has even complex dimension $n = 2m$, it can be considered as a vector space over $\mathbb{H}$, the quaternions and a **quaternionic structure** on $\mathbb{V}$ as an antilinear map $\theta_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}$ with respect to $\mathbf{i} \in \mathbb{H}$, such that $\sigma_{\mathbb{V}}^2 = -\mathrm{Id}_{\mathbb{V}}$.

Example 1.2.2 Let $\mathbb{V} = \mathbb{C}^{2m}$ and

$$\mathrm{J}_m := \begin{pmatrix} 0 & \mathrm{I}_m \\ -\mathrm{I}_m & 0 \end{pmatrix}. \tag{1.2.9}$$

Then $\sigma_{\mathbb{V}}(v) := \mathrm{J}_m \bar{v}$ is a quaterionic structure on $\mathbb{V}$.

Proposition 1.2.1 ([38] p. 41) Let $\langle \cdot , \cdot \rangle$ be an inner product on $\mathbb{V}$. These statements are equivalent:

(1) $\mathbb{V}$ is equipped with a real structure $\sigma_{\mathbb{V}}$.

(2) There is a real vector space $\mathbb{V}'$ such that $\mathbb{V} = \mathbb{V}' \otimes_{\mathbb{R}} \mathbb{C}$.

(3) There is a linear isomorphism $B : \mathbb{V} \to \mathbb{V}^*$ such that $B^t = B$, where B^t denotes the transpose of B.

Proof: (1) $\Rightarrow$ (2) Take $\mathbb{V}' = \mathbb{V}^{\sigma_{\mathbb{V}}}$.

(2) $\Rightarrow$ (1) If $\mathbb{V} = \mathbb{V}' \otimes_{\mathbb{R}} \mathbb{C}$ take $\sigma_{\mathbb{V}}(v', \lambda) = (v', \bar{\lambda})$, for all $v' \in \mathbb{V}, \lambda \in \mathbb{C}$.

(1) $\Rightarrow$ (3) Changing $\langle \cdot , \cdot \rangle$ by $\langle \cdot , \cdot \rangle + \overline{\langle \sigma_{\mathbb{V}} \cdot , \sigma_{\mathbb{V}} \cdot \rangle}$ we may assume that $\langle \sigma_{\mathbb{V}} v, \sigma_{\mathbb{V}} v' \rangle = \langle \bar{v}, \bar{v'} \rangle$, for all $v, v' \in \mathbb{V}$. Now, take $B(v) = \langle \sigma_{\mathbb{V}} v, \cdot \rangle$ for all $v \in \mathbb{V}$.

(3) $\Rightarrow$ (1) Let $\langle \cdot , \cdot \rangle$ be an inner product on $\mathbb{V}$ we define $A : \mathbb{V} \to \mathbb{V}$ by $\langle Av, w \rangle = \Re(B(v, w))$, for every $v, w \in \mathbb{V}$, where $\Re$ denotes the real part of a complex number. Let $A = P\, S$ a polar descomposition, then S is an antilinear involution. $\square$

Remark 1.2.2 A similar proposition can be proved changing real structure by quaterionic structure. In this case, $\mathbb{V} = \mathbb{V}' \otimes_{\mathbb{C}} \mathbb{H}$, and the quaternionic structure here is given by $\theta_{\mathbb{V}}(v', \lambda) = (v', j\lambda)$, for every $v' \in \mathbb{V}'$ and $\lambda \in \mathbb{C}$ where $j \in \mathbb{H}$. The last condition means that there is a linear isomorphism $B : \mathbb{V} \to \mathbb{V}^*$ such that $B^T = -B$.

Remark 1.2.3 From Proposition 1.2.1, we deduce that given a complex vector space $\mathbb{V}$ there always exists a real structure, since every complex vector space is the complexification of a real vector space with the same real dimension. Every antilinear map in a complex vector space is, up to change of basis, the canonical conjugation given by $v \mapsto \bar{v}$, for every $v \in \mathbb{V}$. Two square matrices A, B are consimilar if there is an invertible matrix P such that $A = P\, B, \bar{P}^{-1}$, where $\bar{P}$ is the complex conjugation matrix of P. The set of antilinear maps of $\mathbb{V}$ is the set of consimilar matrices to the conjugation, which is in bijection with $\mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(n, \mathbb{R})$.

Definition 1.2.3 A **real unitary structure** on $\mathbb{V}$ is an antilinear map $\theta_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}^*$ such that $\theta_{\mathbb{V}} = (\theta_{\mathbb{V}}^T)^{-1}$.

Example 1.2.3 Let $\mathbb{V} = \mathbb{C}^n$, let $(v_1, \ldots, v_n)$ be a basis of $\mathbb{C}^n$ and $(\omega_1, \ldots, \omega_n)$ be the dual base. Let $\theta_{\mathbb{V}}(v_i) := \overline{\omega_i}$, where $\overline{\omega_i}(v_j) := \omega_i(\overline{v_j})$ and $1 \leq i, j \leq n$. Then $\theta_{\mathbb{V}}$ is a real unitary structure on $\mathbb{C}^n$.

Example 1.2.4 Let $0 \leq p, q \leq n$ be two natural numbers such that $p + q = n$ and

$$\mathrm{I}_{p,q} := \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & \mathrm{I}_q \end{pmatrix}. \tag{1.2.10}$$

Let $(v_1, \ldots, v_n)$ be a basis of $\mathbb{C}^n$ and $(\omega_1, \ldots, \omega_n)$ be the dual base. Let $\theta_{\mathbb{V}}(v_i) := \mathrm{I}_{p,q}\,\overline{\omega_i}$, for $1 \leq i, j \leq n$, where $\overline{\omega_i}$ is defined as in Example (1.2.4). Then $\theta_{\mathbb{V}}$ is a real unitary structure on $\mathbb{V}$.

1.3 Real structures on Lie algebras and Lie groups

1.3.1 Real structures on Lie algebras

In this section we follow the approach in [42] and [40, Sections 1 and 2].

Definition 1.3.1 A **real form** of a complex Lie algebra $\mathfrak{g}$ is a real subalgebra $\mathfrak{g}_0$ of the underlying real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ such that $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0$.

Definition 1.3.2 A **real structure or conjugation** of a complex Lie algebra $\mathfrak{g}$ is an antilinear involutions of $\mathfrak{g}$.

We denote by $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Conj}(\mathfrak{g})$ the linear algebraic group of all automorphisms and the set of all antilinear involutions of a complex lie algebra $\mathfrak{g}$, respectively. We define the equivalence relation for $\sigma, \sigma' \in \mathrm{Conj}(\mathfrak{g})$ as follows

$$\sigma \sim \sigma' \text{ if there is } \alpha \in \mathrm{Aut}(\mathfrak{g}) \text{ such that } \sigma' = \alpha\sigma\alpha^{-1}.$$

Proposition 1.3.1 There is a bijection between the set of real forms of $\mathfrak{g}$ up to isomorphisms and the quotient space $\mathrm{Conj}(\mathfrak{g})/\sim$.

Proof: Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. We define $\sigma \in \mathrm{Conj}(\mathfrak{g})$ by $\sigma(x + \sqrt{-1}y) = x - \sqrt{-1}y$ for all $x, y \in \mathfrak{g}_0$. Conversely, given a class $[\sigma]$ in $\mathrm{Conj}(\mathfrak{g})/\sim$, the set of fixed points $\mathfrak{g}^\sigma$ of $\mathfrak{g}$ under $\sigma \in \mathrm{Conj}(\mathfrak{g})$ is a real form and if $\sigma' \sim \sigma$, i.e. if there is $\alpha \in \mathrm{Aut}(\mathfrak{g})$ such that $\sigma' = \alpha\sigma\alpha^{-1}$, then $\mathfrak{g}^{\sigma'} = \alpha(\mathfrak{g}^\sigma)$, so $[\sigma]$ induces a real form $\mathfrak{g}^\sigma$ that is unique up to isomorphisms. $\square$

Let τ be the conjugation corresponding to a maximal compact real form of $\mathfrak{g}$. Let $\mathrm{Aut}_2(\mathfrak{g})$ be the set of automorphisms of $\mathfrak{g}$ of order two. As above, we define an equivalence relation $\sim$ in $\mathrm{Aut}_2(\mathfrak{g})$.

Proposition 1.3.2 (Cartan) There is a bijection:

$$\begin{array}{ccc} \mathrm{Conj}(\mathfrak{g})/\sim & \longrightarrow & \mathrm{Aut}_2(\mathfrak{g})/\sim \\ \sigma & \longmapsto & \theta := \sigma \circ \tau\,. \end{array}$$

1.3.2 Real structures on Lie groups

Definition 1.3.3 A **real structure or conjugation** of a complex Lie group G is an antiholomorphic involution of G.

We denote by $\mathrm{Aut}(G)$ and $\mathrm{Conj}(G)$ the group of holomorphic automophisms and the set of antiholomorhic involutions of a complex Lie group G, respectively.

Definition 1.3.4 A **real form** of G is a real Lie subgroup G_0 of the underlying real group $G_{\mathbb{R}}$ of G such that G_0 is the fixed point set on G of some $\sigma \in \mathrm{Conj}(G)$.

Two conjugations are related $\sigma \sim \sigma'$ if and only if there is $\alpha \in \mathrm{Aut}(G)$ such that $\sigma' = \alpha \circ \sigma \circ \alpha^{-1}$. This is an equivalence relation and two equivalent conjugations define the same real form of G up to isomorphisms. The set of equivalence classes of real forms of G is in bijection with $\mathrm{Conj}(G)/\sim$. We can define similarly the equivalence relation $\sim$ in the group $\mathrm{Aut}_2(G)$ of automophisms of G of order 2. Let $\tau \in \mathrm{Conj}(G)$ be a conjugation such that the fixed point set G^τ of G under τ is a maximal compact subgroup of G.

Proposition 1.3.3 (Cartan) There is a bijection:

$$\begin{array}{ccc} \mathrm{Conj}(G)/\sim & \longrightarrow & \mathrm{Aut}_2(G)/\sim \\ \sigma & \longmapsto & \theta := \sigma \circ \tau\,. \end{array}$$

Corollary 1.3.5 Let G be a connected reductive Lie group equipped with an antiholomorphic involution σ. There is a maximal compact subgroup $H \subset G$ that is invariant under σ.

Proof: By the theorem of Cartan there is a unique antiholomorphic involution τ defining a maximal compact real form H of G such that $\sigma \circ \tau = \tau \circ \sigma$, so H is invariant under σ. □

Remark 1.3.1 An explicit construction of H can be done following [15, p. 3]. Let $\widehat{G}$ be the group $G \times (\mathbb{Z}/2\mathbb{Z})$ as a set, with the group operation

$$(g_1, e_1)(g_2, e_2) = (g_1(\sigma_G)^{e_1}(g_2), e_1 + e_2)\,.$$

Take a maximal compact subgroup $\widehat{H}$ of $\widehat{G}$ containing the element $(Id_G, Id_{\mathbb{Z}/2})$. It exists because this element has order two and the group it generates is compact, so is contained in a maximal compact subgroup. The subgroup H is $\widehat{H} \cap G$.

Definition 1.3.6 The **adjoint action** $\mathrm{Ad} : G \to \mathrm{Aut}(G)$ is the action of G on G given by

$$\mathrm{Ad}(g)g' = g^{-1}\, g'\, g, \quad \text{for all } g, g' \in G\,.$$

Definition 1.3.7 An **inner automorphism of** G is an element of $\mathrm{Aut}(G)$ of the form $\mathrm{Ad}(g)$, for $g \in G$.

We denote by $\mathrm{Int}(G)$ the normal subgroup of $\mathrm{Aut}(G)$ of all inner automophisms. Two elements $\sigma, \sigma' \in \mathrm{Aut}_2(G)$ are related $\sigma \sim_i \sigma'$ by an inner automophism if and only if there is $\mathrm{Ad}(g) \in \mathrm{Int}(G)$ such that $\sigma' = \mathrm{Ad}(g) \circ \sigma$. Let $\mathrm{Out}_2(G) = \mathrm{Aut}_2(G)/\sim_i$ be the outer automorphism group of elements of order two. Let

$$\mathfrak{cl} : \mathrm{Aut}_2(G) \to \mathrm{Out}_2(G)$$

be the natural projection.

Definition 1.3.8 The **adjoint representation** $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the linear action of G on its Lie algebra $\mathfrak{g}$ given by the diferential at the identity of the adjoint action 1.3.6, i.e. $\mathrm{Ad}(g) := d_{\mathrm{Id}_G}(\mathrm{Ad}(g)), \quad \text{for all } g \in G$.

Definition 1.3.9 The adjoint group $\mathrm{Ad}(G)$ of G is the image of G under the adjoint representation.

We fix $a \in \mathrm{Out}_2(G)$. Let $\theta \in \mathfrak{cl}^{-1}(a)$. Since $\theta \in \mathrm{Aut}_2(G)$, then it defines by restriction an action on $\mathrm{Ad}(G)$. We call to this action θ again. We define as a set of 1-cocycles the elements of $\mathrm{Ad}(G)$. Two cocycles $a, a' \in \mathrm{Ad}(G)$ are θ-cohomologous if there is $b \in \mathrm{Ad}(G)$ satisfying

$$a' = b^{-1} a \theta(b)\,.$$

We denote by $H^1_\theta(\mathbb{Z}/2, \mathrm{Ad}(G))$ the cohomology group consisting of θ-cohomologous classes of 1-cocycles. Let $\theta' \in \mathfrak{cl}^{-1}(a)$ be another automorphism in the antiimage by $\mathfrak{cl}$ of a, by [40, Proposition 2.7, Proposition 2.8] there is a bijection between $H^1_\theta(\mathbb{Z}/2, \mathrm{Ad}(G))$ and $H^1_{\theta'}(\mathbb{Z}/2, \mathrm{Ad}(G))$, so it has sense to denote to this cohomology by $H^1_a(\mathbb{Z}/2, \mathrm{Ad}(G))$.

Proposition 1.3.4 ([40], Proposition 2.9) Let $a \in \mathrm{Out}_2(G)$. There is a bijection between $\mathfrak{cl}^{-1}(a)$ and $H^1_a(\mathbb{Z}/2, \mathrm{Ad}(G))$.

1.3.3 Compatible conjugations of real forms

Definition 1.3.10 Let G be a real form of a reductive Lie group $G^{\mathbb{C}}$ and $\mu \in \mathrm{Conj}(G^{\mathbb{C}})$ be the conjugation such that the fixed point set $(G^{\mathbb{C}})^\mu$ of $G^{\mathbb{C}}$ under μ is equal to G. Let $\tau \in \mathrm{Conj}(G^{\mathbb{C}})$ such that $(G^{\mathbb{C}})^\tau = H^{\mathbb{C}}$ is a maximal compact subgroup of $G^{\mathbb{C}}$. A conjugation $\sigma_G \in \mathrm{Conj}(G^{\mathbb{C}})$ is (μ, τ)-**compatible** if

(1) $\sigma_G \circ \mu = \mu \circ \sigma_G$.

(2) $\sigma_G \circ \tau = \tau \circ \sigma_G$.

Proposition 1.3.5 Let G be a real form of a reductive Lie group $G^{\mathbb{C}}$ and σ_G be (μ, τ)-compatible conjugation of $G^{\mathbb{C}}$. Then:

(1) $\theta := \sigma \circ \tau \in \mathrm{Aut}(G^{\mathbb{C}})$ satisfies $G^\theta = H$.

(2) $G = (G, H, d\theta, B)$ is a real reductive Lie group in the sense of Knapp [55, p. 384] where B is the Killing bilinear form on the Lie algebra $\mathfrak{g}$.

(3) $\sigma_G(G) = G$.

(4) $\sigma_G(H) = H$.

Proof:

(1) It is a consequence of $(G^{\mathbb{C}})^\tau = H^{\mathbb{C}}$, $(G^{\mathbb{C}})^\mu = G$ and $\tau = \theta \circ \mu$.

(2) By (1), $d\theta$ is a Cartan involution of $\mathfrak{g}$.

(3) It is a consequence of $(G^{\mathbb{C}})^\mu = G$ and condition (1) of the Definition 1.3.10.

(4) It is a consequence of (3) and condition (2) of the Definition 1.3.10. $\square$

1.4 Real structures on G-spaces

Definition 1.4.1 A real structure σ_M on a complex manifold M is an antiholomorhic involution σ_M on M.

Definition 1.4.2 Let G be a complex Lie group equipped with a real structure σ_G. Let M be a complex manifold equipped with a real structrure σ_M. An action of G on M is (σ_G, σ_M)-**real** if

$$\sigma_G(g)\sigma_M(m) = \sigma_M(gm)\,, \text{ for every } g \in G \text{ and } m \in M\,.$$

Example 1.4.1 Let G be a complex Lie group equipped with a real structure σ_G. The adjoint action $\mathrm{Ad} : G \to \mathrm{Aut}(G)$ defined in 1.3.6 is a (σ_G, σ_G)-real action. Let $\mathfrak{g}$ be the Lie algebra of G. Since $d_{\mathrm{Id}_G} \circ \sigma_G = d\sigma_G \circ\ d_{\mathrm{Id}_G}$, then the adjoint action $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ defined in 1.3.8 is a $(\sigma_G, d\sigma_G)$-real action. Finally, the action $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ defined by $\mathrm{Ad} := d_{\mathrm{Id}_G}\mathrm{Ad}$ is also a $(d\sigma_G, d\sigma_G)$-real action.

Definition 1.4.3 Let G be a complex group equipped with a real structure σ_G. Let $\mathbb{V}$ be a complex vector space equipped with a real structure $\sigma_{\mathbb{V}}$ (a quaternionic structure $\sigma_{\mathbb{V}}$). A representation $\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$ is $(\sigma_G, \sigma_{\mathbb{V}})$-**real** (resp. $(\sigma_G, \sigma_{\mathbb{V}})$-**quaternionic**) if for every $g \in G$ and $v \in \mathbb{V}$,

$$\sigma_{\mathbb{V}}(\,\rho(g)v\,) = \rho(\,\sigma_G(g)\,)\sigma_{\mathbb{V}}(v)\,.$$

A real structure $\sigma_{\mathbb{V}}$ (resp. quatenionic structure $\sigma_{\mathbb{V}}$) induces the following antiholomorphic involution $\sigma_{\mathrm{GL}(\mathbb{V})}$ in $\mathrm{GL}(\mathbb{V})$:

$$\sigma_{\mathrm{GL}(\mathbb{V})}(A) = \sigma_{\mathbb{V}} \circ A \circ \sigma_{\mathbb{V}}^{-1}, \text{ for all } A \in \mathrm{GL}(\mathbb{V})\,.$$

A representation $\rho : G \to \mathrm{GL}(\mathbb{V})$ is $(\sigma_G, \sigma_{\mathbb{V}})$-real (resp. $(\sigma_G, \sigma_{\mathbb{V}})$-quaternionic) if and only if the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\sigma_G"'] & \mathrm{GL}(\mathbb{V}) \arrow[d, "\sigma_{\mathrm{GL}(\mathbb{V})}"] \\ G \arrow[r, "\rho"] & \mathrm{GL}(\mathbb{V})\,. \end{tikzcd}$$

Remark 1.4.1 A $(\sigma_G, \sigma_{\mathbb{V}})$-real representation $\rho : G \to \mathrm{GL}(\mathbb{V})$ is a linear action of G on $\mathbb{V}$ that is $(\sigma_G, \sigma_{\mathbb{V}})$-real.

Example 1.4.2 Let $G = \mathrm{GL}(n, \mathbb{C})$ and $\sigma_G(A) := \bar{A}$ be the conjugate matrix of $A \in \mathrm{GL}(n, \mathbb{C})$. Let $\mathbb{V} = \mathbb{C}^n$ and $\sigma_{\mathbb{V}}$ be the complex conjugation (See Example 1.2.1.) The fundamental representation $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(\mathbb{V})$ is a $(\sigma_G, \sigma_{\mathbb{V}})$-real representation.

Example 1.4.3 Let $G = \mathrm{GL}(2m, \mathbb{C})$ and $\sigma_G(A) := \mathrm{J}_m \circ \bar{A} \circ \mathrm{J}_m^{-1}$, where $\bar{A}$ is the conjugate matrix and J_m is defined in (1.2.9). Let $\mathbb{V} = \mathbb{C}^{2m}$ and $\sigma_{\mathbb{V}}$ be the quaternionic structure given in Example 1.2.2. The fundamental representation $\mathrm{GL}(2m, \mathbb{C}) \to \mathrm{GL}(\mathbb{V})$ is a $(\sigma_G, \sigma_{\mathbb{V}})$-quaternionic representation.

Definition 1.4.4 Let G be a complex group equipped with a real structure σ_G. Let $\mathbb{V}$ be a complex vector space equipped with a real unitary structure $\theta_{\mathbb{V}}$. A representation $\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$ is $(\sigma_G, \theta_{\mathbb{V}})$-**real unitary** if for every $g \in G$ and $v \in \mathbb{V}$,

$$\theta_{\mathbb{V}}(\rho(g)v) = \rho^*(\sigma_G(g))\theta_{\mathbb{V}}(v)\,,$$

where $\rho^* : G \to \mathrm{GL}(\mathbb{V}^*)$ is the dual representation of ρ defined as follows: $\rho^*(g) := \rho(g^{-1})^T$, for every $g \in G$.

A real unitary structure $\theta_{\mathbb{V}}$ induces an antiholomorphic map $\theta_{\mathrm{GL}(\mathbb{V})} : \mathrm{GL}(\mathbb{V}) \to \mathrm{GL}(\mathbb{V}^*)$ defined by $\theta_{\mathrm{GL}(\mathbb{V})}(A) = \theta_{\mathbb{V}} \circ A \circ \theta_{\mathbb{V}}^T$, for all $A \in \mathrm{GL}(\mathbb{V})$. A representation $\rho : G \to \mathrm{GL}(\mathbb{V})$ is $(\sigma_G, \sigma_{\mathbb{V}})$-real unitary if and only if the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\sigma_G"'] & \mathrm{GL}(\mathbb{V}) \arrow[d, "\theta_{\mathrm{GL}(\mathbb{V})}"] \\ G \arrow[r, "\rho^*"] & \mathrm{GL}(\mathbb{V}^*)\,. \end{tikzcd}$$

Example 1.4.4 Let $G = \mathrm{GL}(n, \mathbb{C})$ and $\sigma_G(A) := (\bar{A}^T)^{-1}$ be conjugate inverse transpose of $A \in \mathrm{GL}(n, \mathbb{C})$. Let $\mathbb{V} = \mathbb{C}^n$ and $\sigma_{\mathbb{V}}$ be the real unitary structure given in Example 1.2.3. The fundamental representation $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(\mathbb{V})$ is a $(\sigma_G, \sigma_{\mathbb{V}})$-real unitary representation.

Example 1.4.5 Let $G = \mathrm{GL}(n, \mathbb{C})$ and $\sigma_G(A) := \mathrm{I}_{p,q}(\bar{A}^T)^{-1}\mathrm{I}_{p,q}$, for any matrix $A \in \mathrm{GL}(n, \mathbb{C})$. Let $\mathbb{V} = \mathbb{C}^n$ and $\sigma_{\mathbb{V}}$ be the real unitary structure given by Example

1.2.4. The fundamental representation $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(\mathbb{V})$ is a $(\sigma_G, \sigma_{\mathbb{V}})$-real unitary representation.

1.5 Real structures on vector bundles

Definition 1.5.1 Let X be a complex manifold equipped with a a real structure σ_X and let $E \to X$ be a holomorphic vector bundle. A σ_X-**real structure** $\sigma_E : E \to E$ on E is a lift of σ_X,

$$\begin{tikzcd} E \arrow[r, "\sigma_E"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\sigma_X"] & X \end{tikzcd}$$

satisfying:

(1) σ_E is antiholomorhic,

(2) σ_E is antilinear in the fibres,

(3) $\sigma_E^2 = \mathrm{Id}_E$.

If we change property (3) above by $\sigma_E^2 = -\mathrm{Id}_E$, then σ_E is called a σ_X-**quaternionic structure on** E. Let E and E' be two holomorphic vector bundles over X equipped with real (quaternionic) structrures σ_E and $\sigma_{E'}$, respectively.

Definition 1.5.2 A morphism of vector bundles $f : E \to E'$ is $(\sigma_E, \sigma_{E'})$-**real** (resp. $(\sigma_E, \sigma_{E'})$-**quaternionic**) if f satisfies

$$f \circ \sigma_E = \sigma_{E'} \circ f\,.$$

Definition 1.5.3 A σ_X-**real unitary structure** $\theta_E : E \to E^*$ on E is a lift of σ_X,

$$\begin{tikzcd} E \arrow[r, "\theta_E"] \arrow[d] & E^* \arrow[d] \\ X \arrow[r, "\sigma_X"] & X \end{tikzcd}$$

satisfying:

(1) θ_E is antiholomorhic,

(2) θ_E is antilinear in the fibres,

(3) $\theta_E \circ \theta_E^T = \mathrm{Id}_E$, where θ_E^T is the dual morphism of θ_E .

Definition 1.5.4 Let E and E' be two holomorphic vector bundles over X equipped with real unitary structrures θ_E and $\theta_{E'}$, respectively. A morphism of vector bundles $f : E \to E'$ is $(\theta_E, \theta_{E'})$-**real unitary** if f satisfies

$$\theta_E = f^T \circ \theta_{E'} \circ f \,,$$

where $f^T : (E')^* \to E^*$ is defined by the transpose in fibres of f, i.e. $(f^T)_x := (f_x)^T$.

Let E be a holomorphic vector bundle and let ψ be a real structure, or a quaternionic structrure or a real unitary structrure, a pair (E, ψ) will be referred as a **real bundle** or **quaternionic bundle** or **real unitary bundle** resp. This terminology may be misleading since by a real bundle, quaternionic bundle and unitary bundle, we usually mean a bundle whose transition functions take values on $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{H})$ and $\mathrm{U}(p, q)$, resp. But we hope it is clear from the context. These two notions of reality are related in the following way, if σ_X has fixed points, a real or quaternionic or real unitary structure ψ defines a real or quaterionic or unitary bundle E^ψ over X^{σ_X} in the sense that its transition functions take values on $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{H})$ and $U(p, q)$, resp. In [72, p. 7], [4, p. 370] the authors give a proof of this relation for real structures and algebraic varieties:

Theorem 1.5.5 Let X be a complex algebraic variety and σ_X a real structure such that the set of fixed points $X^{\sigma_X} \neq \emptyset$. There is an equivalence of categories between holomorphic bundles over X equipped with a real structure and real morphisms and the category of algebraic real vector bundles over a real manifold $X^{\sigma_X} = \partial(X)$ with linear homomorphisms.

1.5.1 Dolbeaut operators and real structures

Let X be a complex manifold and $\mathbb{E}$ be a C^∞ complex vector bundle over X. We denote by $\Omega^0(\mathbb{E})$ the set of C^∞ sections of $\mathbb{E}$ and $\Omega^{0,1}(\mathbb{E})$ to the C^∞ sections of $TX^{(0,1)} \otimes \mathbb{E}$. Let p and q be natural numbers, we denote by $\Omega^{p,q}(\mathbb{E})$ the C^∞ sections of $\Lambda^p TX^{(1,0)} \otimes \Lambda^q TX^{(0,1)} \otimes \mathbb{E}$.

A Dolbeaut operator is a $\mathbb{C}$-linear operator

$$\bar{\partial}_E : \Omega^0(\mathbb{E}) \longrightarrow \Omega^{0,1}(\mathbb{E})$$

satisfying the Leibniz rule: $\bar{\partial}_E(fs) = \bar{\partial} f \otimes s + f\bar{\partial}_E(s)$, for all $f \in C^\infty(X)$ and $s \in \Omega^0(\mathbb{E})$. One can extend the definition of Dolbeaut operator to $\bar{\partial}_E : \Omega^{0,q}(\mathbb{E}) \longrightarrow \Omega^{0,q+1}(\mathbb{E})$ by

$$\bar{\partial}_E(\omega \otimes s) := \bar{\partial}(w) \otimes s + (-1)^q \omega \wedge \partial_E(s),$$

for any ω section of $\Lambda^q TX^{(0,1)}$ and $s \in \Omega^0(\mathbb{E})$. A Dolbeaut operator is called integrable if $\bar{\partial}_E^2 = \bar{\partial}_E \circ \bar{\partial}_E = 0$.

Definition 1.5.6 Let σ_X be a real structure on X. A σ_X-**real structure** on C^∞ complex vector bundle $\mathbb{E}$ over X is a map $\sigma_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}$, such that lifts the involution σ_X ,

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\sigma_{\mathbb{E}}"] \arrow[d] & \mathbb{E} \arrow[d] \\ X \arrow[r, "\sigma_X"] & X \end{tikzcd}$$

and satisfies:

(1) $\sigma_{\mathbb{E}}$ is antilinear in the fibres,

(2) $\sigma_{\mathbb{E}}^2 = \mathrm{Id}_{\mathbb{E}}$.

If we change property (2) above by $\sigma_{\mathbb{E}}^2 = -\mathrm{Id}_{\mathbb{E}}$, then $\sigma_{\mathbb{E}}$ is called a σ_X-**quatenionic structure** on $\mathbb{E}$. A real (quatenionic) structure $\sigma_{\mathbb{E}}$ on $\mathbb{E}$ defines an (anti)involution $\sigma_{\Omega^0(\mathbb{E})}$ on $\Omega^0(\mathbb{E})$ given by

$$\sigma_{\Omega^0(\mathbb{E})}(s)(\sigma_X(x)) := \sigma_{\mathbb{E}}(s(x))\,,$$

for every $s \in \Omega^0(\mathbb{E})$ and $x \in X$ and also defines a map $\sigma_{\Omega^{0,1}(\mathbb{E})} : \Omega^{0,1}(\mathbb{E}) \to \Omega^{0,1}(\mathbb{E})$ given by

$$\sigma_{\Omega^{0,1}(\mathbb{E})}(s)(\sigma_X(x)) = d\sigma_X \otimes \sigma_{\mathbb{E}}(s(x))\,,$$

for all $s \in \Omega^{0,1}(\mathbb{E})$ and $x \in X$.

Definition 1.5.7 A Dolbeaut operator $\bar{\partial}_E$ on $\mathbb{E}$ is $(\sigma_X, \sigma_{\mathbb{E}})$-**real** (resp. $(\sigma_X, \sigma_{\mathbb{E}})$-**quaternionic**) if the following diagram commutes

$$\begin{tikzcd}
\Omega^0(\mathbb{E}) \arrow[r, "\bar{\partial}_E"] \arrow[d, "\sigma_{\Omega^0(\mathbb{E})}"] & \Omega^{0,1}(\mathbb{E}) \arrow[d, "\sigma_{\Omega^{0,1}(\mathbb{E})}"] \\
\Omega^0(\mathbb{E}) \arrow[r, "\bar{\partial}_E"] & \Omega^{1,0}(\mathbb{E}) .
\end{tikzcd}$$

Theorem 1.5.8 Let $\mathbb{E}$ be a C^∞ bundle over X equipped with a σ_X-real structure $\sigma_{\mathbb{E}}$ (σ_X-quaternionic structure $\sigma_{\mathbb{E}}$). There is a bijection between real bundles (quaternionic bundles) with underlying C^∞ bundle $\mathbb{E}$ and $(\sigma_X, \sigma_{\mathbb{E}})$-real (quaternionic) integrable Dolbeaut operators $\bar{\partial}_E$ on $\mathbb{E}$.

Proof: Let rank$(\mathbb{E}) = r$. Recall that a C^∞ complex bundle $\mathbb{E}$ is holomorphic if and only if for every $x_0 \in X$, there is an open neighborhood U and r holomorphic sections $\{s_i\}_{1\leq i\leq r}$ of $\mathbb{E}$ over U, such that $\{s_i(x)\}_{1\leq i\leq k}$ form a basis of $\mathbb{E}_x$ varying $x \in U$. This condition happens if and only if $\mathbb{E}$ is equipped with an integrable Dolbeaut operator that annihilates these sections. (See [60, p.18]).

A σ_X-real (σ_X-quaternionic) structure $\sigma_{\mathbb{E}}$ defines a σ_X-real (σ_X-quaternionic) structure σ_E antiholomorhic on E if and only if for every $x_0 \in X$, there exist an open neighborhood U, and r holomorphic sections $\{s_i\}_{1\leq i\leq r}$ satisfying $\sigma_{\Omega^0}(s_i) = s_i$, for all $x \in U$. The integrable Dolbeaut operator which annihilates this sections satisfies a reality condition which is precisely to be $(\sigma_X, \sigma_{\mathbb{E}})$-real ($(\sigma_X, \sigma_{\mathbb{E}})$-quaternionic). □

Let $\bar{\partial}_E$ a Dolbeaut operator on $\mathbb{E}$, we define the **associated Dolbeaut operator** $\bar{\partial}_{E^*}$ on $\mathbb{E}^*$, as follows:

$$(\bar{\partial}_{E^*})_X(s^*)(s) = \bar{\partial}(s^*s)(X) - s^*((\bar{\partial}_E)_X(s))\,, \tag{1.5.11}$$

for all $X \in \mathrm{T}X, s^x \in \Omega^0(\mathbb{E}^*)$ and $v \in \Omega^0(\mathbb{E})$.

Lemma 1.5.9 If $\bar{\partial}_E$ is a $(\sigma_X, \sigma_{\mathbb{E}})$-real (quaternionic) Dolbeaut operator on $\mathbb{E}$, then $\bar{\partial}_{E^*}$ is a $(\sigma_X, \sigma_{\mathbb{E}^*})$-real (quaternionic) Dolbeaut operator on $\mathbb{E}^*$, where $\sigma_{\mathbb{E}^*} := \sigma_{\mathbb{E}}^T$.

Proof: We have to prove that

$$(\bar{\partial}_{E^*})_{\bar{X}} \circ \sigma_{\mathbb{E}^*} = \sigma_{\mathbb{E}^*} \circ (\bar{\partial}_{E^*})_X\,. \tag{1.5.12}$$

The left hand side of (1.5.12) is equal to:

$$(\bar{\partial}_{E^*})_{\bar{X}} \circ \sigma_{\mathbb{E}^*} s^*(s) = \bar{\partial}(s^* \sigma_{\mathbb{E}} s)X - \sigma_{\mathbb{E}^*} s^*(\,(\bar{\partial}_E)_{\bar{X}} s\,) = \bar{\partial}(\sigma_{\mathbb{E}^*} s^* s)X - s^*(\,(\bar{\partial}_E)_{\bar{X}} \sigma_{\mathbb{E}} s\,)\,, \tag{1.5.13}$$

for all $s \in \Omega^0(\mathbb{E})$ and $s^* \in \Omega^0(\mathbb{E}^*)$. The right hand side of (1.5.12) is equal to

$$\sigma_{\mathbb{E}^*} \circ (\bar{\partial}_{E^*})_{\bar{X}} s^*(s) = \bar{\partial}(\,\sigma_{\mathbb{E}^*} s^*(s)\,)X - s^*(\,\sigma_{\mathbb{E}}(\bar{\partial}_E)_X s\,)\,, \tag{1.5.14}$$

for all $s \in \Omega^0(\mathbb{E})$ and $s^* \in \Omega^0(\mathbb{E}^*)$. Since $\bar{\partial}_E$ is a $(\sigma_X, \sigma_{\mathbb{E}})$-real (quaternionic) Dolbeaut operator on $\mathbb{E}$, then

$$(\bar{\partial}_E)_{\bar{X}} \circ \sigma_{\mathbb{E}} = \sigma_{\mathbb{E}} \circ (\bar{\partial}_E)_X\,,$$

so (1.5.13) is equal to (1.5.14) and (1.5.12) holds.

□

Definition 1.5.10 Let X be a complex manifold equipped with a real structure σ_X and let $\mathbb{E}$ be a C^∞ complex vector bundle over X. A σ_X-**real unitary structure** on $\mathbb{E}$ is a map $\theta_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}^*$, such that lifts the involution σ_X, i.e.

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\theta_{\mathbb{E}}"] \arrow[d] & \mathbb{E}^* \arrow[d] \\ X \arrow[r, "\sigma_X"] & X \end{tikzcd}$$

and satisfies:

(1) $\theta_{\mathbb{E}}$ is antilinear in the fibres,

(2) $\theta_{\mathbb{E}}^T \circ \theta_{\mathbb{E}} = \mathrm{Id}_{\mathbb{E}}$.

A real unitary structure $\theta_{\mathbb{E}}$ on $\mathbb{E}$ defines a map $\theta_{\Omega^0(\mathbb{E})} : \Omega^0(\mathbb{E}) \to \Omega^0(\mathbb{E}^*)$ by

$$\theta_{\Omega^0(\mathbb{E})}(s)(\sigma_X(x)) := \theta_{\mathbb{E}}(s(x))\,,$$

for every $s \in \Omega^0(\mathbb{E})$ and $x \in X$ and also defines a map $\theta_{\Omega^{0,1}(\mathbb{E})} : \Omega^{0,1}(\mathbb{E}) \to \Omega^{0,1}(\mathbb{E}^*)$ given by

$$\theta_{\Omega^{0,1}(\mathbb{E})}(s)(\sigma_X(x)) = d\sigma_X \otimes \theta_{\mathbb{E}}(s(x))\,,$$

for all $s \in \Omega^{0,1}(\mathbb{E})$ and $x \in X$.

Definition 1.5.11 A Dolbeault operator $\bar{\partial}_E$ on $\mathbb{E}$ is $(\sigma_X, \theta_{\mathbb{E}})$-**real unitary** if the following diagram commutes

$$\begin{tikzcd} \Omega^0(\mathbb{E}) \arrow[r, "\bar{\partial}_E"] \arrow[d, "\theta_{\Omega^0(\mathbb{E})}"] & \Omega^{0,1}(\mathbb{E}) \arrow[d, "\theta_{\Omega^{0,1}(\mathbb{E})}."] \\ \Omega^0(\mathbb{E}^*) \arrow[r, "\bar{\partial}_{E^*}"] & \Omega^{0,1}(\mathbb{E}^*)\,. \end{tikzcd}$$

Theorem 1.5.12 Let $\mathbb{E}$ be a C^∞ bundle over X equipped with a σ_X-real unitary structure $\theta_{\mathbb{E}}$. There is a bijection between real unitary bundles with underlying C^∞ bundle $\mathbb{E}$ and $(\sigma_X, \theta_{\mathbb{E}})$-real unitary integrable Dolbeault operators $\bar{\partial}_E$ on $\mathbb{E}\,,$.

Proof: Same proof that in Theorem 1.5.8. $\square$

1.5.2 Compatible Hermitian metrics

Definition 1.5.13 Let X be a complex manifold equipped with a real structure σ_X and let $\mathbb{E}$ be a C^∞ complex bundle equipped with a σ_X-real (quaterionic) structure $\sigma_{\mathbb{E}}$. A Hermitian metric $h : \mathbb{E} \to \mathbb{E}^*$ is $(\sigma_X, \sigma_{\mathbb{E}})$-**compatible** if the following diagram commutes

$$\begin{tikzcd} \mathbb{E} \arrow[r, "h"] \arrow[d, "\sigma_{\mathbb{E}}"] & \mathbb{E}^* \arrow[d, "\sigma_{\mathbb{E}^*}"] \\ \mathbb{E} \arrow[r, "h"] & \mathbb{E}^*\,, \end{tikzcd} \tag{1.5.15}$$

where $\sigma_{\mathbb{E}^*} := \sigma_{\mathbb{E}}^T$ is the transpose of $\sigma_{\mathbb{E}}$.

Definition 1.5.14 Let $\mathbb{E}$ be a C^∞ complex bundle equipped with a σ_X-real unitary structure $\theta_{\mathbb{E}}$. A Hermitian metric $h : \mathbb{E} \to \mathbb{E}^*$ is $(\sigma_X, \theta_{\mathbb{E}})$-**compatible** if the following diagram commutes

$$\begin{tikzcd} \mathbb{E} \arrow[r, "h"] \arrow[d, "\theta_{\mathbb{E}}"] & \mathbb{E}^* \arrow[d, "\theta_{\mathbb{E}^*}"] \\ \mathbb{E}^* \arrow[r, "h^{-1}"] & \mathbb{E}\,, \end{tikzcd} \tag{1.5.16}$$

where $\theta_{\mathbb{E}^*} := \theta_{\mathbb{E}}^T$.

1.5.3 Compatible connections

Let X be a complex manifold and $\mathbb{E}$ a C^∞ complex vector bundle over X. Let j be a natural number and let $\Omega^j(\mathbb{E})$ be the C^∞ sections of $\Lambda^j TX^* \otimes \Omega^0(\mathbb{E})$. A connection

D on $\mathbb{E}$ is a $\mathbb{C}$–linear differential operator

$$D : \Omega^0(\mathbb{E}) \longrightarrow \Omega^1(\mathbb{E})$$

satisfying the Leibniz rule

$$D(fs) = df \otimes s + f\, D(s)\,,$$

for all $f \in C^\infty(X), s \in \Omega^0(\mathbb{E})$. One can extend the definition of connection to $D : \Omega^j(\mathbb{E}) \longrightarrow \Omega^{j+1}(\mathbb{E})$ by

$$D(\omega \otimes s) := d(w) \otimes s + (-1)^j \omega \wedge D(s)\,,$$

for any ω section of $\Lambda^j TX^*$ and $s \in \Omega^0(\mathbb{E})$.

Definition 1.5.15 Let σ_X be a real structure on X and $\sigma_{\mathbb{E}}$ be a σ_X-real structure (quaternionic structure) on $\mathbb{E}$. On $\Omega^1(\mathbb{E})$ we can define the (anti)involution $\sigma_{\Omega^1(\mathbb{E})} := d\sigma_X \otimes \sigma_{\Omega^0(\mathbb{E})}$. A connection D on $\mathbb{E}$ is $(\sigma_X, \sigma_{\mathbb{E}})$-**compatible** if the following diagram commutes

$$\begin{tikzcd} \Omega^0(\mathbb{E}) \arrow[r, "D"] \arrow[d, "\sigma_{\Omega^0(\mathbb{E})}"] & \Omega^1(\mathbb{E}) \arrow[d, "\sigma_{\Omega^1(\mathbb{E})}"] \\ \Omega^0(\mathbb{E}) \arrow[r, "D"] & \Omega^1(\mathbb{E})\,. \end{tikzcd} \tag{1.5.17}$$

Definition 1.5.16 Let h be a Hermitian structure on $\mathbb{E}$. A connection D is called h-**connection** if

$$d(h(e, e')) = h(De, e') + h(e, De')\,,$$

for all $e, e' \in \Omega^0(\mathbb{E})$.

Theorem 1.5.17 Let $\mathbb{E}$ be a C^∞-vector bundle over X equipped with a σ_X-real (quaternionic) structure $\sigma_{\mathbb{E}}$. Fix an $(\sigma_X, \sigma_{\mathbb{E}})$-compatible Hermitian metric h on $\mathbb{E}$. There is a bijective correspondence between $(\sigma_X, \sigma_{\mathbb{E}})$-real (quaternionic) Dolbeaut operators and $(\sigma_X, \sigma_{\mathbb{E}})$-compatible h-connections.

Proof: There is a bijective correspondence between Dolbeaut operators and h-connections. Given an h-connection, the $(0,1)$-part defines a Dolbeaut operator and conversely given a Dolbeaut operator $\bar{\partial}_E$ on $\mathbb{E}$ the Chern connection associated is defined by the sum of $\bar{\partial}_E$ and

$$\partial_h := h^{-1} \circ \bar{\partial}_{E^*} \circ h \,, \tag{1.5.18}$$

where $\bar{\partial}_{E^*}$ is the associated Dolbeaut operator defined in 1.5.11. We only have to prove that ∂_h satisfies the following condition for every $X \in \mathrm{T}X$

$$(\partial_h)_{\bar{X}} \circ \sigma_{\mathbb{E}} = \sigma_{\mathbb{E}} \circ (\partial_h)_X \,. \tag{1.5.19}$$

Indeed, by (1.5.18) this equality is equivalent to

$$h^{-1} \circ (\bar{\partial}_{E^*})_{\bar{X}} \circ h \circ \sigma_{\mathbb{E}} = \sigma_{\mathbb{E}} \circ h^{-1} \circ (\bar{\partial}_{E^*})_X \circ h \,.$$

Since h is a $(\sigma_X, \sigma_{\mathbb{E}})$-compatible, then we have

$$h^{-1} \circ (\bar{\partial}_{E^*})_{\bar{X}} \circ \sigma_{\mathbb{E}^*} \circ h = h^{-1} \circ \sigma_{\mathbb{E}^*} \circ (\bar{\partial}_{E^*})_X \circ h \,.$$

and finally this equality is true because by Lemma 1.5.9, one has that $\bar{\partial}_{E^*}$ is a $(\sigma_X, \sigma_{\mathbb{E}^*})$-real (quaternionic) Dolbeaut operator on $\mathbb{E}^*$. □

Definition 1.5.18 Let X be a complex manifold equipped with a real structure σ_X on X and let $\mathbb{E}$ be a C^∞ complex vector bundle over X equipped with a θ_X-real unitary structure on $\mathbb{E}$. On $\Omega^1(\mathbb{E})$ there is a map $\theta_{\Omega^1} : \Omega^1(\mathbb{E}) \to \Omega^1(\mathbb{E}^*)$ given by $d\sigma_X \otimes \theta_{\Omega^0(\mathbb{E})}$. A connection D on $\mathbb{E}$ is $(\sigma_X, \theta_{\mathbb{E}})$-**compatible** if the following diagram commutes

$$\begin{tikzcd} \Omega^0(\mathbb{E}) \arrow[r, "D"] \arrow[d, "\theta_{\Omega^0(\mathbb{E})}"] & \Omega^1(\mathbb{E}) \arrow[d, "\theta_{\Omega^1(\mathbb{E})}"] \\ \Omega^0(\mathbb{E}^*) \arrow[r, "D^*"] & \Omega^1(\mathbb{E}^*) \,, \end{tikzcd} \tag{1.5.20}$$

where D^* is the associated connection on $\mathbb{E}^*$ defined by

$$(D^*)_X(s^*)(s) = X(s^* s) - s^*(D(s)) \,, \tag{1.5.21}$$

for all $X \in \mathrm{T}X, s^* \in \Omega^0(\mathbb{E}^*)$ and $s \in \Omega^0(\mathbb{E})$.

Theorem 1.5.19 Let $\mathbb{E}$ be a C^∞-vector bundle over X equipped with a σ_X-real unitary structure $\theta_{\mathbb{E}}$. Fix an $(\sigma_X, \sigma_{\mathbb{E}})$-compatible Hermitian metric h on $\mathbb{E}$. There is a bijective correspondence between $(\sigma_X, \sigma_{\mathbb{E}})$-real unitary Dolbeaut operators and $(\sigma_X, \theta_{\mathbb{E}})$-compatible h-connections.

Proof: The proof is similar to the proof of Theorem 1.5.17. □

1.5.4 Topological classification

Theorem 1.5.20 Let (X, σ_X) be a Klein surface with topological parameters (g, k, a) and E a holomorphic complex vector bundle of $\deg d$ and rank r.

A σ_X-real structure σ_E on E exists:

- for $k = 0$, if and only if d is even, or zero.
- if $k > 0$, let $X^{\sigma_X} \approx \coprod_{i=1}^{k} \gamma_i$ be descomposition of the fixed points in ovals, where γ_i is a homotopic to a circle and $(w(1), ..., w(k))$ be the set of the first Stiefel-Whitney class of $E^{\sigma_E} \to X^\sigma$ restricted to γ_j, then

$$w(1) + ... + w(n) = \, d \bmod 2\,.$$

A σ_X-quaternionic structure σ_E on E exists and it is unique if and only if

$$d + r(g-1) = 0 \mod 2\,.$$

A σ_X-real unitary structure σ_E on E exists and it is unique if and only if $d = 0$.

Proof: The real and quaternionic cases are proved in [17, p.8]. For the real unitary case, recall that in general, the topological classification of G-bundles over S^1 is given by $\pi_0(G)$. In the unitary case this group is trivial. □

1.6 Real structures on principal bundles

Let G be a reductive complex Lie group equipped with a real structure σ_G, and let X be a complex manifold equipped with a real structure σ_X. Let Z be the center of G and $Z_2^{\sigma_G}$ the subgroup of elements of Z of order two, that are invariant under σ_G.

Let $c \in Z_2^{\sigma_G}$. We can suppose that c belongs to this set (see [17, Proposition 3.8]). Let E be a holomorphic principal G-bundle over X.

Definition 1.6.1 A map $\sigma_E : E \to E$ is a (σ_X, σ_G, c)-**real structure** if it is a lift of σ_X

$$\begin{tikzcd} E \arrow[r, "\sigma_E"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\sigma_X"] & X, \end{tikzcd}$$

satisfying:

(1) σ_E is antiholomorphic,

(2) $\sigma_E(eg) = \sigma_E(e)\,\sigma_G(g)$, for all $e \in E$, $g \in G$,

(3) $\sigma_E^2(e) = ce$.

The pair (E, σ_E) will be referred as a (σ_X, σ_G, c)-real G-bundle.

Proposition 1.6.1 The set of (σ_X, σ_G, c)-real structures σ_E on a G-bundle E is in one-to-one correspondence with the set of holomorphic bijections $\phi_E : E \to \sigma_X^* E \times_{\sigma_G} G$ such that $\sigma_X^* \sigma_G(\phi_E) \circ \phi_E = c\,\mathrm{Id}_E$.

Proof: Let $f : E \to \sigma_X^* E \times_{\sigma_G} G$ be a lift of σ_X given by

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & \sigma_X^* E \arrow[r, "\pi"] \arrow[d] & \sigma_X^* E \times_{\sigma_G} G \arrow[d] \\ X \arrow[r, "\sigma_X"] & X \arrow[r, "\mathrm{Id}"] & X, \end{tikzcd}$$

where $\pi(e) := (e, \mathrm{Id}_G)$, for all $e \in E$. If σ_E is a (σ_X, σ_G, c)-real structure, then $\phi_E := f \circ \sigma_E$ is a holomorphic isomorphism of G-bundles and it satisfies $\phi_E^2(\mathrm{Id}_E) = \sigma_E^2(\mathrm{Id}_E) = c$.

Conversely, let $f^{-1} : \sigma_X^* E \times_{\sigma_G} G \to E$ a lift of σ_X given by

$$\begin{tikzcd} \sigma_X^* E \times_{\sigma_G} G \arrow[r, "p"] \arrow[d] \arrow[rr, bend left, "f^{-1}"] & \sigma_X^* E \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\mathrm{Id}"] & X \arrow[r, "\sigma_X"] & X, \end{tikzcd}$$

where $p(e, g) := e\sigma_G(g)$, for all $e \in \sigma_X^* E$. If $\phi_E : E \to \sigma_X^* \sigma_G(E)$ is a holomorphic isomorphism such that $\sigma_X^* \sigma_G(\phi_E) \circ \phi_E = c \operatorname{Id}_E$, then $\sigma_E := f^{-1} \circ \phi_E$ is a (σ_X, σ_G, c)-real structure. □

1.6.1 Holomorphic structures and real structures.

Definition 1.6.2 Let G be a reductive complex Lie group and let X be a complex manifold equipped with real structures σ_G and σ_X, respectively. Let $\pi : \mathbb{E} \to X$ be a C^∞ principal G-bundle over X. Let $c \in Z_2^{\sigma_G}$. A (σ_X, σ_G, c)-**real structure** on $\mathbb{E}$ is a map $\sigma_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}$, which lifts the involution σ_X,

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\sigma_{\mathbb{E}}} & \mathbb{E} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma_X} & X \end{array}$$

and satisfies:

(1) $\sigma_{\mathbb{E}}(ge) = \sigma_G(g)\, \sigma_E(e)$, for all $e \in E$, and $g \in G$,

(2) $\sigma_{\mathbb{E}}^2(e) = c\, e$.

We denote by $\mathcal{C}$ the set of G-invariant complex structures on $\pi : \mathbb{E} \to X$ such that $d\pi : T\mathbb{E} \to \pi^* TX$ is a complex map.

Definition 1.6.3 Let $\sigma_{\mathbb{E}}$ be a (σ_X, σ_G, c)-real structure on $\mathbb{E}$. A complex structure $J_{\mathbb{E}} \in \mathcal{C}$ is $\sigma_{\mathbb{E}}$-**real** if

$$J_{\mathbb{E}} \circ d\sigma_{\mathbb{E}} \;=\; -d\sigma_{\mathbb{E}} \circ J_{\mathbb{E}} \tag{1.6.22}$$

and $d\pi$ is a $(d\sigma_{\mathbb{E}}, \sigma_X)$-compatible morphism.

We denote by $\mathcal{C}(\mathbb{E}, \sigma_{\mathbb{E}})$ the set of G-invariant complex structures that are $\sigma_{\mathbb{E}}$-real.

Proposition 1.6.2 Let E be a principal G-bundle. There a bijection between the set of $(\sigma_X, \sigma_G, \operatorname{Id}_E)$-real G-bundles (E, σ_E) and $\mathcal{C}(\mathbb{E}, \sigma_E)$, where $\mathbb{E}$ is the C^∞ underlying G-bundle.

Proof: Equation (1.6.22) holds if and only if $\sigma_{\mathbb{E}}$ is antiholomorhic. □

1.6.2 Compatible connections and compatible reductions

Let G be a reductive complex Lie group and let X be a complex manifold equipped with real structures σ_G and σ_X, respectively. Let $\pi : \mathbb{E} \to X$ be a C^∞ principal G-bundle over X. Let $\sigma_{\mathbb{E}}$ be a (σ_X, σ_G, c)-real structure on $\mathbb{E}$.

Proposition 1.6.3 A connection on $\mathbb{E}$ is said to be $\sigma_{\mathbb{E}}$-**compatible** if it is one of the following equivalent objects:

(1) An equivariant distribution H such that $T\mathbb{E}_G = V\mathbb{E}_G \oplus H$ and

$$(\sigma_{\mathbb{E}})_* H_e = H_{\sigma_{\mathbb{E}}(e)}\,,$$

for all $e \in \mathbb{E}$.

(2) An equivariant section Γ of $V\mathbb{E}_G \to T\mathbb{E}_G$, such that the following diagram commutes

$$\begin{tikzcd} V\mathbb{E} \arrow[r] \arrow[d, "d\sigma_{\mathbb{E}}"] & T\mathbb{E} \arrow[l, bend right, "\Gamma"'] \arrow[d, "d\sigma_{\mathbb{E}}"] \\ V\mathbb{E} \arrow[r] & T\mathbb{E}\,. \end{tikzcd}$$

(3) A 1-form $w \in \Omega^1(\mathbb{E}, \mathfrak{g})$, such that the map $w : T_y\mathbb{E}_G \to \mathfrak{g}$ is $(\sigma_{T\mathbb{E}_G}, d\sigma_G)$-real, where $\sigma_{T\mathbb{E}_G}$ is the involution of $T\mathbb{E}_G$ defined by $\sigma_{\mathbb{E}}$.

(4) A differential operator D on $\mathbb{E}$ with values on $\mathfrak{g}$ satisfing:

$$d(\sigma_G) \circ D \circ \sigma_{\mathbb{E}} = D\,.$$

Proof: (1) $\Rightarrow$ (2) Let $X = X_v \oplus X_h$ be the descomposition of an element

$$X \in T\mathbb{E} = V\mathbb{E} \oplus H,$$

where this descomposition is given by hypothesis. Using the fact that H is real we obtain

$$d\sigma_{\mathbb{E}}(X_h) = \Gamma(d\sigma_{\mathbb{E}}(X_h)) + d\sigma_{\mathbb{E}}(X_v))\,.$$

So we have

$$d\sigma_{\mathbb{E}} \circ \Gamma = \Gamma \circ d\sigma_{\mathbb{E}}$$

and this equality proves the statement.

(2) $\Rightarrow$ (3) The isomorphism $\mathbb{E} \times \mathfrak{g} \to V\mathbb{E}$ is given by $(y, A) \to A^*_y$. If we have an equivariant section Γ and $\Gamma(X) = A^*_y$ then

$$\Gamma(d\sigma_{\mathbb{E}}(X)) = d\sigma_{\mathbb{E}}(A^*_y) = (d\sigma_G A)^*_{\sigma_G(y)}\,,$$

so we conclude

$$\sigma^*_{\mathbb{E}} w(X) = w(d\sigma_{\mathbb{E}}(X)) = d\sigma_G w(X).$$

(3) $\Rightarrow$ (1) If $X \in H_e = \mathrm{Ker}(w)$, since w is real, then

$$w_{\sigma_{\mathbb{E}} e}(d\sigma_{\mathbb{E}}(X)) = d\sigma_G w(X) = 0$$

and this implies that $d\sigma_{\mathbb{E}} X \in H_{\sigma_{\mathbb{E}}(e)} = \mathrm{Ker}(w_{\sigma_{\mathbb{E}}(e)})$.

(3) $\Leftrightarrow$ (4) We can define $D = d + w$. The operator D satisfies $d(\sigma_G) \circ D \circ \sigma_F = D$ if and only if w is real. $\square$

Let $\mathcal{A}$ be the space of connections on $\mathbb{E}$ and let $\mathcal{A}^{1,1}$ be the subset of $\mathcal{A}$ consisting of connections whose curvatures belongs to $\Omega^{1,1}(\,\mathbb{E}_H(\mathfrak{h})\,)$. We denote by $\mathcal{A}^{1,1}(\mathbb{E}, \sigma_{\mathbb{E}})$ the subset of $\mathcal{A}^{1,1}$ consisting of $\sigma_{\mathbb{E}}$-compatible connections.

Definition 1.6.4 Let $\mathbb{E}$ be C^∞ principal G-bundle over X equipped with a (σ_X, σ_G, c)-real structure $\sigma_{\mathbb{E}}$. Let $H \subset G$ a maximal compact subgroup invariant by σ_G. (see Corollary 1.3.5). A reduction of the structure group $\mathbb{E}_H$ of $\mathbb{E}$ from G to H is $\sigma_{\mathbb{E}}$-**compatible** if

$$\sigma_{\mathbb{E}}(\mathbb{E}_H) = \mathbb{E}_H\,.$$

Proposition 1.6.4 Let $\mathbb{E}_K \subset \mathbb{E}$ be a $\sigma_{\mathbb{E}}$-compatible reduction. There is a bijection between $\mathcal{A}^{1,1}(\mathbb{E}_H, \sigma_{\mathbb{E}}|_{\mathbb{E}_H})$ and $\mathcal{C}(\mathbb{E}, \sigma_{\mathbb{E}})$

Proof: The Chern map $C : \mathcal{C} \to \mathcal{A}^{1,1}$ defined by

$$C(J) = H := J(T(E_K)) \cap T(E_K)$$

establishes a bijection (See [61, Lemma 2.1]). If J is complex structure $\sigma_{\mathbb{E}}$-real, then $C(J)$ is a connection on $\mathbb{E}_H$ $\sigma_{\mathbb{E}}|_{\mathbb{E}_H}$-compatible, because using (3) of Proposition 1.6.3

$$H_{\sigma_E(e)} = J(T_{\sigma_E(e)}(E_K)) \cap T_{\sigma_E(e)}(E_K) = d\sigma_E(H_e)\,.$$

$\square$

1.6.3 Topological classification

Let G be a reductive complex Lie group with a real structure σ_G, let (X, σ_X) be a Klein surface of topological type (g, k, a) and let $\{\gamma_i\}_{1 \leq i \leq k}$ the k ovals of X^{σ_X}.

Fix $c \in H^2(\mathbb{Z}/2\mathbb{Z}, Z)$. Let E be a principal G-bundle equipped with a (σ_G, σ_X, c)-real structure σ_E. For each γ_i we choose a class $\alpha_i \in H^1_c(\mathbb{Z}/2\mathbb{Z}, G)$ such that the image of the class h is c. This is the topological type of the bundle over each point of γ_i. Let h_{α_i} representing the class α_i in $H^1_c(\mathbb{Z}/2\mathbb{Z}, G)$.

The following definition and classificacion Theorem are given in [17, p.13]. The generalized **Stiefel-Whitney class** for the topological type α_i over γ_i is an element in $\pi_0(\mathrm{Stab}(h_{\alpha_i}))$.

Proposition 1.6.1 *The topological classes of (σ_X, σ_G, c)-real G-bundles over a Klein surface are classified by:*

(1) Classes $\alpha_i \in H^1_c(\mathbb{Z}/2\mathbb{Z}, G)$ for each oval γ_i.

(2) Generalized Stiefel-Whitney classes for the topological type α_i for each γ_i.

(3) Homotopy classes of equivariant maps of X in the classifying space BG mapping the ovals γ_i to the generalized Stiefel-Whitney classes.

1.7 Real structures on associated bundles

Definition 1.7.1 Let X be a complex manifold equipped with a real structure σ_X and G be a complex reductive group equipped with a real structure σ_G. Let $c \in Z_2^{\sigma_G}$, E be a complex G-bundle over X equipped with a a (σ_G, σ_X, c)-real structure and M be a (σ_G, σ_M)-real G-space such that $cm = m$ for all $m \in M$.

Let $E(M)$ be the quotient bundle $E \times M$ by the equivalence relation given by $(eg, g^{-1}m) = (e, m)$, for any $e \in E, m \in M, g \in G$. The **associated antiholomorphic involution** on $E(M)$ is

$$\sigma_{E(M)}(e, m) := (\sigma_E(e), \sigma_M(m)) .$$

Example 1.7.1 (A real structure on associated vector bundles)

Let $G = \mathrm{GL}(n, \mathbb{C})$ and $\sigma_G(A) := \bar{A}$ be the conjugate matrix, for any $A \in \mathrm{GL}(n, \mathbb{C})$. Let $M = \mathbb{C}^n$ and σ_M be the complex conjugation (See Example 1.2.1). The vector space M is a (σ_G, σ_M)-real G-space with the action given by the fundamental representation (See Example 1.4.2). Take $c = \mathrm{Id}_{\mathrm{GL}(n,\mathbb{C})}$, then $V = E(M)$ is a vector bundle and $\sigma_V = \sigma_{E(M)}$ is antilinear in the fibres so σ_V is a σ_X-real structure on V .

Example 1.7.2 (A real structure on the adjoint bundle)

Let G be a reductive complex Lie group equipped with a real structure σ_G. Let $M = G$ and $\sigma_M = \sigma_G$. The group G is a (G, σ_G)-real G-space because the adjoint action is (σ_G, σ_G)-real (See Example 1.4.1). Take any $c \in Z_2^{\sigma_G}$, then $\mathrm{Ad}(E)$ is a G-bundle, and $\sigma_{E(M)} = \sigma_{\mathrm{Ad}(E)}$.

Example 1.7.3 (A real structure on the Lie algebra of the adjoint bundle) Let G be a reductive complex Lie group equipped with a real structure σ_G. Let $M = \mathfrak{g}$ the Lie algebra of G and $\sigma_M = d\sigma_G$. The Lie algebra $\mathfrak{g}$ is a $(\sigma_G, d\sigma_G)$-real G-space with the action given by the Lie algebra of the adjoint representation $G \to \mathrm{GL}(\mathfrak{g})$ (See Example 1.4.1). Take any $c \in Z_2^{\sigma_G}$, then $V = E(\mathbb{V}) = \mathrm{ad}(E)$ is a vector bundle, and $\sigma_V = \sigma_{\mathrm{ad}(E)}$ is antilinear in the fibres so it is σ_X-real structure on $\mathrm{ad}(E)$.

The following proposition will be useful in the next chapter. There is a one-to-one correspondence between sections of $E(M)$ and antiequivariant maps from E to M. We extend this result to the case in which all objects have real structures.

Proposition 1.7.1 Real sections of $(E(M), \sigma_{E(M)})$ are in one-to-one correspondence with (σ_E, σ_M)-real G-equivariant morphisms $\phi_s : E \to M$.

Proof: Let s be a section of $E(M)$ and $\phi_s : E \to M$ be the antiequivariant map associated to s. We recall that

$$\phi_s(e) = m \leftrightarrow s(x) = (e, m)\,, \tag{1.7.23}$$

for every $e \in E$ and $x \in E_x$. It is suffice to prove that the antiequivariant map $\phi_{\sigma_{E(M)} \circ s \circ \sigma_X}$ associated to $\sigma_{E(M)} \circ s \circ \sigma_X$ satisfies

$$\phi_{\sigma_{E(M)} \circ s \circ \sigma_X} = \sigma_M \circ \phi_s \circ \sigma_E\,, \tag{1.7.24}$$

because one has that s is a real section if and only if $s = \sigma_{E(M)} \circ s \circ \sigma_X$ and by (1.7.24) this happens if and only if $\phi_s = \sigma_M \circ \phi_s \circ \sigma_E$, i.e. ϕ_s is a (σ_E, σ_M)-real G-equivariant morphism. Let $m' = \phi_s(\sigma_E(e))$, by (1.7.23) one has

$$s(\,\sigma_X(x)\,) \;=\; (\sigma_E(e), m')\,. \tag{1.7.25}$$

On the other hand, Equation (1.7.24) is equivalent to

$$\phi_{\sigma_{E(M)} \circ s \circ \sigma_X} = \sigma_M(m')\,. \tag{1.7.26}$$

By (1.7.23), the last equation is equivalent to

$$\sigma_{E(M)} \circ s \circ \sigma_X(x) = (e, \sigma_M(m'))\,,$$

but this equality is true, because

$$\sigma_{E(M)} \circ s \circ \sigma_X(x) = \sigma_{E(M)}(\sigma_E(e), m') = (e, \sigma_M(m'))\,,$$

where the first equality holds by Equation (1.7.25) and the second by definition of $\sigma_{E(M)}$. □

Let X be a complex manifold equipped with a real structure σ_X and G be a complex reductive group equipped with a real structure σ_G. Let $c \in Z_2^{\sigma_G}$, E be a complex G-bundle over X a with a a (σ_G, σ_X, c)-real structure and M a complex vector space equipped with a quaternionic (unitary) real structure σ_M. Let $\rho : G \to GL(M)$ be a (σ_G, σ_M)-quaternionic (unitary) representation.

Let $E(M)$ be the quotient bundle $E \times_\rho M$ by the equivalence relation given by $(eg, \rho(g^{-1})m) = (e, m)$ for any $e \in E, m \in M, g \in G$. and let $E(\sigma_M(M))$ be the quotient bundle $E \times \sigma_M(M)$ by the equivalence relation given by $(\,e\sigma_G(g), \sigma_M(\rho(g^{-1})m)\,) = (e, m)$ for any $e \in E, m \in M, g \in G$. We define an antiholomorhic map

$$\sigma_{E(M)} : E(M) \to E(\sigma_M(M))$$

by $\sigma_{E(M)}(e, m) := (\sigma_E(e), \sigma_M(m))$.

Example 1.7.4 (A quaternionic structure on associated vector bundles)

Let $G = \mathrm{GL}(2m, \mathbb{C})$ and $\sigma_G(A) := \mathrm{J}_m \circ \bar{A} \circ \mathrm{J}_m^{-1}$. Let $M = \mathbb{C}^{2m}$ and σ_M be the quaternionic structure given in Example 1.2.2. The fundamental representation $\mathrm{GL}(2m, \mathbb{C}) \to \mathrm{GL}(M)$ is a (σ_G, σ_M)-quaternionic representation (See Example

1.4.3). Take $c = -\mathrm{Id}_{\mathrm{GL}(2m,\mathbb{C})}$, and let V be the vector bundle $E(M)$, then $\sigma_V = \sigma_{E(M)}$ is antilinear in the fibres and therefore, σ_V is a σ_X-quaternionic structure on V.

Example 1.7.5 (Real structures on associated $\mathrm{U}(p,q)$-bundles)

Let $G = \mathrm{GL}(n,\mathbb{C})$ and σ_G be the map sending $A \mapsto \mathrm{I}_{p,q}(\bar{A}^T)^{-1}\mathrm{I}_{p,q}$, for any $A \in \mathrm{GL}(n,\mathbb{C})$. Let $M = \mathbb{C}^n$ and $\theta_M : M \to M^*$ be the real unitary structure given in Example 1.2.4. The fundamental representation $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(\mathbb{V})$ is a $(\sigma_G, \sigma_{\mathbb{V}})$-real unitary representation (See Example 1.4.5). Take $c = \mathrm{Id}_{\mathrm{GL}(n,\mathbb{C})}$ and let V be the vector bundle $E(M)$, then $\sigma_V = \sigma_{E(M)}$ is antilinear in the fibres, so σ_V is a σ_X-real unitary structure on V.

Chapter 2

Hitchin-Kobayashi correspondence for real Higgs pairs

Along this chapter, we use the following notation:

- $(X, \sigma_X)-$compact Klein surface
- $G-$connected complex reductive Lie group
- $\mathfrak{g}-$Lie algebra of G
- σ_G-real structure on G
- $H-$maximal compact Lie subgroup of G invariant under σ_G
- $\mathfrak{h}-$Lie algebra of H
- $B : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}-$non degenarate bi-invariant metric on $\mathfrak{h}$
- $\mathbb{V}-$complex vector space
- $\sigma_{\mathbb{V}}$ real (quaternionic) structure on $\mathbb{V}$
- $\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$ – $(\sigma_G, \sigma_{\mathbb{V}})$-real (quaternionic) representation
- $\langle \cdot, \cdot \rangle$ – H-invariant Hermitian product on $\mathbb{V}$ compatible with $\sigma_{\mathbb{V}}$
- $Z-$center of G

- $\mathfrak{z}$−Lie algebra of Z
- $\mathfrak{z}(\mathfrak{h})$−center of $\mathfrak{h}$
- $Z_2^{\sigma_G}$−subgroup of elements of Z of order two that are invariants under σ_G.

2.1 Real structures on Higgs pairs

First, we review the definition of Higgs pairs, following [39]. Let X be a compact Riemann surface and G be a connected reductive complex Lie group. Let $\mathbb{V}$ be a complex vector space and $\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$ be a representation. We denote by $V := E(\mathbb{V})$ the vector bundle associated to E via ρ. Let L be a line bundle over X.

Definition 2.1.1 *L***-Higgs pairs** are pairs (E, φ) consisting of a G-bundle E over X and a holomorphic section φ of $V \otimes L$.

Let σ_X be a real structure on X and σ_G be a real structure on G. Let $\sigma_{\mathbb{V}}$ be a real or a quaternionic structure on $\mathbb{V}$. Let $\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$ be a $(\sigma_G, \sigma_{\mathbb{V}})$-real or quaterionic representation, respectively. (See Section 1.4). Let Z be the center of G and $Z_2^{\sigma_G}$ be the subgroup of elements of Z of order two that are invariants under σ_G. Let $c \in Z_2^{\sigma_G} \cap \ker \rho$. Let (E, σ_E) be a (σ_X, σ_G, c)-real G-bundle over (X, σ_X) (Definition 1.6.1). Since $c \in \mathrm{Ker}(\rho)$, the antiholomorphic map $\sigma_V : V \to V$ defined for all $(e, v) \in V$ by

$$\sigma_V(e, v) := (\sigma_E(e), \sigma_{\mathbb{V}}(v))$$

is a real (quaterionic) structure on V. Let (L, σ_L) be a real or quaternionic line bundle over (X, σ_X). The tensor product $\sigma_V \otimes \sigma_L$ defines a real structure on $V \otimes L$ if and only if σ_V and σ_L are both real or quaternionic structures. The tensor product $\sigma_V \otimes \sigma_L$ defines a quaternionic structure on $V \otimes L$ if and only if one of the structures σ_V, σ_L is real and the other is quaternionic.

Definition 2.1.2 A $(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)$**-real (quaternionic) structure** on a Higgs pair (E, φ) is an antiholomorphic map $\sigma_E : E \to E$ such that (E, σ_E) is a (σ_X, σ_G, c)-real (quaternionic) G-bundle over (X, σ_X) and

$$\sigma_V \otimes \sigma_L(\varphi) = \pm\sigma_X^* \varphi. \tag{2.1.1}$$

The triple (E, φ, σ_E) will be referred as a $(\sigma_X, \sigma_G, c, \sigma_L, \pm)$**-real (quaternionic) Higgs pair** and if there is no confusion as a $(c, \sigma_L, \pm)$**-real (quaternionic) Higgs pair.**

Let E^L be the $(G \times \mathbb{C}^*)$-bundle given by

$$E^L = \{(e, l) \in E \times_X L,\ l \neq 0\}\,, \tag{2.1.2}$$

where, by abuse of notation, L is identified with its correspondening principal $\mathbb{C}^*$-bundle. We define

$$\sigma_{E^L}(e, l) := (\sigma_E(e), \mathrm{conj}(l))\,,$$

for every $e \in E$ and $l \neq 0 \in L$, where conj denotes an antilinear conjugation on L. The pair (E^L, σ_{E^L}) is a $(\sigma_X, \sigma_G \times \mathrm{conj}, c)$-real $(G \times \mathbb{C}^*)$-bundle.

Lemma 2.1.3 Let $\tilde{\rho} : G \times \mathbb{C}^* \to \mathrm{GL}(\mathbb{V})$ be the representation defined by

$$\tilde{\rho}(g, \lambda)(v) := \lambda \rho(g) v\,,$$

for all $g \in G\,, v \in \mathbb{V}$ and $\lambda \in \mathbb{C}^*$. If ρ is a $(\sigma_G, \sigma_{\mathbb{V}})$-real (quaterionic) representation, then $\tilde{\rho}$ is a $(\sigma_G \times \mathrm{conj}, \sigma_{\mathbb{V}})$-real (quaterionic) representation, respectively.

Proof: For every $g \in G, v \in \mathbb{V}$ and $\lambda \in \mathbb{C}^*$,

$$\tilde{\rho}(\sigma_G \times \mathrm{conj}(g, \lambda))\sigma_{\mathbb{V}}(v) = \mathrm{conj}(\lambda)\rho(\sigma_G g)(\sigma_{\mathbb{V}}(v)) = \mathrm{conj}(\lambda)\sigma_{\mathbb{V}}(\rho(g)(v)) = \sigma_{\mathbb{V}}(\tilde{\rho}(g, \lambda)v)\,.$$

The first equality holds by definition of $\tilde{\rho}$, the second because ρ is real (quaternionic) and the third because $\sigma_{\mathbb{V}}$ is antilinear. $\square$

The associated bundle $E^L(\mathbb{V})$ to E^L via $\tilde{\rho}$ is equipped with a real (quaternionic) structure $\sigma_{E^L(\mathbb{V})}$. (See Section 1.7).

Proposition 2.1.1 The real (quaternionic) vector bundle $(E^L(\mathbb{V}), \sigma_{E^L(\mathbb{V})})$ is isomorphic to $(V \otimes L, \sigma_V \otimes \sigma_L)$.

Proof: Consider the isomophism of vector bundles

$$\begin{array}{ccc} V \otimes L & \xrightarrow{T} & E^L(\mathbb{V}) \\ ((e, v), l) & \longmapsto & ((e, \lambda), v)\,, \end{array}$$

where λ is the element associated to $l \in L$ as a $\mathbb{C}^*$-bundle. For all $e \in E$, $v \in \mathbb{V}$ and $l \in L$, one has

$$T(\,\sigma_V \otimes \sigma_L((e,v),l)\,) = T(\,(\sigma_E(e),\sigma_{\mathbb{V}}(v)),\sigma_L(l)\,) = (\,(\sigma_E(e),\mathrm{conj}(\lambda)),\sigma_{\mathbb{V}}(v)\,)$$
$$=\sigma_{E^L} \otimes \sigma_{\mathbb{V}}(\,(e,\lambda),v\,) = \sigma_{E^L} \otimes \sigma_{\mathbb{V}}(T((e,v),l)))\,.$$

Therefore, T is a $(\sigma_V \otimes \sigma_L, \sigma_{E^L(\mathbb{V})})$-real isomorphism of real vector bundles (Definition 1.5.2). □

Finally, we give some examples of real and quaternionic Higgs pairs.

Example 2.1.1 (Pseudo-real G-Higgs bundles)

- (G, σ_G) any reductive complex Lie group with an antiholomorphic involution
- $(L, \sigma_L) = (K, \sigma_K)$ the canonical bundle with antiholomorphic involution induced by σ_X
- $(\mathbb{V}, \sigma_{\mathbb{V}}) = (\mathfrak{g}, d\sigma_G)$, the Lie algebra of G with the antilinear involution coming from σ_G
- $\rho = \mathrm{Ad}$, the adjoint representation is a $(\sigma_G, \sigma_{\mathbb{V}})$-real representation. (Example 1.7.2).

Example 2.1.2 (Quaternionic pairs)

- $(G, \sigma_G) = (\mathrm{GL}(2m, \mathbb{C}), \sigma_G(A) = \mathrm{J}_m \circ \bar{A} \circ \mathrm{J}_m^{-1})$, see the definition of J_m in Example 1.2.2
- (L, σ_L) a real line bundle
- $\mathbb{V} = \mathbb{C}^{2m}$ and $\sigma_{\mathbb{V}})$ with is the quaternionic structure given in Example 1.2.2
- The fundamental representation $\mathrm{GL}(2m, \mathbb{C}) \to \mathrm{GL}(M)$ is a (σ_G, σ_M)-quaternionic representation. (Example 1.4.3)

Example 2.1.3 (Real quadratic bundles)

- $(\mathrm{GL}(n, \mathbb{C}), \mathrm{conj})$, where $\mathrm{conj} : A \mapsto \bar{A}$, for every $A \in \mathrm{GL}(n)$

- $(L, \sigma_L) = (\mathcal{O}_X, \sigma_{\mathcal{O}_X})$, where $\mathcal{O}_X$ is the trivial line bundle and $\sigma_{\mathcal{O}_X}$ is the real structure of $\mathcal{O}_X$ coming from σ_X and the complex conjugation of $\mathbb{C}$

- $(\mathbb{V}, \sigma_{\mathbb{V}}) = (\mathrm{Sym}^2(\mathbb{C}^n), \mathrm{conj})$, where $\mathrm{conj} : v \mapsto \bar{v}$, for every $v \in \mathbb{V}$

- $\rho(v_1, v_2) = (\rho'(v_1), \rho'(v_2))$, for any $(v_1, v_2) \in \mathrm{Sym}^2(\mathbb{C}^n)$, where ρ' the fundamental representation.

Example 2.1.4 (Real triples)

- $(G, \sigma_G) = (\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}), \mathrm{conj} \times \mathrm{conj})$, where $\mathrm{conj}(A)$ is the conjugate matrix of A

- $(L, \sigma_L) = (\mathcal{O}_X, \sigma_{\mathcal{O}_X})$, where $\mathcal{O}_X$ is the trivial line bundle and $\sigma_{\mathcal{O}_X}$ is the real structure of $\mathcal{O}_X$ coming from σ_X and the complex conjugation of $\mathbb{C}$

- $(\mathbb{V}, \sigma_{\mathbb{V}}) = (\mathrm{Hom}(\mathbb{C}^{n_2}, \mathbb{C}^{n_1}), \mathrm{conj})$, where $\mathrm{conj}(T)(\bar{v}) = \overline{Tv}$, for every $T \in \mathrm{Hom}(\mathbb{C}^{n_2}, \mathbb{C}^{n_1})$ and $v \in \mathbb{C}^{n_2}$

- $\rho(A, B)(T) = A \circ T \circ B^{-1}$, for all $(A, B) \in G$ and $T \in \mathbb{V}$.

Example 2.1.5 (Quaternionic triples)

- $(G, \sigma_G) = (\mathrm{GL}(2m, \mathbb{C}) \times \mathrm{GL}(2m, \mathbb{C}), \mathrm{conj}' \times \mathrm{conj}')$, where $\mathrm{conj}' : A \mapsto \mathrm{J}_m \circ \bar{A} \circ \mathrm{J}_m^{-1}$, for every $A \in \mathrm{GL}(2m, \mathbb{C})$

- $(L, \sigma_L) = (\mathcal{O}_X, \sigma_{\mathcal{O}_X})$

- $(\mathbb{V}, \sigma_{\mathbb{V}}) = (\mathrm{Hom}(\mathbb{C}^{2m}, \mathbb{C}^{2m}), \mathrm{conj}')$, where $\mathrm{conj}'(T) = \mathrm{J}_m \circ \bar{T} \circ \mathrm{J}_m^{-1}$, for every $T \in \mathrm{Hom}(\mathbb{C}^{2m}, \mathbb{C}^{2m})$

- $\rho(A, B)(T) = A \circ T \circ B^{-1}$, for all $(A, B) \in G$ and $T \in \mathbb{V}$.

2.2 Hermite-Einstein-Higgs equation

Let G be a reductive complex Lie group equipped with a real structure σ_G. Let $H \subset G$ be a maximal compact Lie subgroup invariant under σ_G. It always exists, see Corollary 1.3.5. Fix a non degenarate bi-invariant metric B on $\mathfrak{h} = \mathrm{Lie}(H)$

compatible with $d\sigma_G$, i.e. $(d\sigma_G)^* B = B$. Let (X, σ_X) be a compact Klein surface. Let ω be a σ_X-real Kähler form on X. (Definition 1.1.7). Let (L, σ_L) be a real (quaterionic) line bundle over (X, σ_X) and fix (σ_X, σ_L)-compatible Hermitian metric h_L on L. We denote by F_L the curvature of the corresponding Chern connection. Let $\mathbb{V}$ a complex space equipped with a real (quaternionic) structure $\sigma_{\mathbb{V}}$. Let $\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$ be a $(\sigma_G, \sigma_{\mathbb{V}})$-real (quaternionic) representation. Let $\langle \cdot, \cdot \rangle$ be a H-invariant Hermitian product on $\mathbb{V}$ compatible with $\sigma_{\mathbb{V}}$, meaning that for all $v, v' \in \mathbb{V}$

$$\langle \sigma_{\mathbb{V}}(v), \sigma_{\mathbb{V}}(v') \rangle = \langle v, v' \rangle .$$

Let h be a σ_E-compatible reduction of the structure group of E from G to H. (See 1.6.4). We denote by E_H the reduced H-bundle induced by h. Let A_h be the unique h-connection on E_H compatible with the holomorphic structure of E. Let F_h be the curvature of A_h, then $F_h \in \Omega^2(E_H(\mathfrak{h}))$. Let Λ be the contraction with ω, then $\Lambda(F_h) \in \Omega^0(E_H(\mathfrak{h}))$.

The vector bundle $V := E(\mathbb{V})$ has a Hermitian metric h_V coming from h and $\langle \cdot, \cdot \rangle$. Since $\langle \cdot, \cdot \rangle$ is H-invariant and compatible with $\sigma_{\mathbb{V}}$, the Hermitian metric h_V is (σ_X, σ_V)-compatible. (See Section 1.5.2). Using h_V and h_L, we obtain a $(\sigma_X, \sigma_V \otimes \sigma_L)$-compatible Hermitian metric $h_{V \otimes L} := h_V \otimes h_L$ on the product $E(\mathbb{V}) \otimes L$. The representation ρ induces a map

$$\begin{tikzcd} U(\mathbb{V})^* \arrow[r, "(d\rho)^*"] \arrow[rr, bend left, "Q"] & \mathfrak{h}^* \arrow[r, "B"] & \mathfrak{h} \end{tikzcd} ,$$

that gives a morphism of vector bundles $Q : E_H(\mathrm{U}(\mathbb{V}))^* \to E_H(\mathfrak{h})$. We consider

$$\mu_h(\varphi) := Q(-\frac{\mathbf{i}}{2} \varphi \otimes \varphi^*) , \tag{2.2.3}$$

where φ^* means the image of φ by $h_{V \otimes L}$, that once we have fixed h_L, depends on h.

Definition 2.2.1 Let (E, φ, σ_E) a $(c, \sigma_L, \pm)$-real (quaternionic) Higgs pair. Let $\mathfrak{z}(\mathfrak{h})$ be the center of the Lie algebra $\mathfrak{h}$. Let $\alpha \in \mathfrak{z}(\mathfrak{h})$ such that $d\sigma_G(\alpha) = -\alpha$. A σ_E-compatible reduction h of the structure group of E from G to H is called **Hermite-Einstein-Higgs** if it satisfies

$$\Lambda(F_h) + \mu_h(\varphi) = -\sqrt{-1}\, \alpha . \tag{2.2.4}$$

Remark 2.2.1 In Section 2.4 we give a a moment map interpretation of Equation (2.2.4).

2.3 Stability conditions

Let G be a reductive complex Lie group, H be a maximal compact subgroup of G and B be a non-degenerate bi-invariant metric on $\mathfrak{h}$.

Definition 2.3.1 Let s be an element $\sqrt{-1}\mathfrak{h}$, the following objects are associated to s:

- $P_s := \{g \in G,\ e^{ts}ge^{-ts}$ is bounded when $t \to \infty\}$ is a parabolic subgroup of G
- $L_s := \{g \in G,\ \mathrm{Ad}(g)s = s\}$ is the Levi subgroup of P_s, where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the Adjoint representation (Definition 1.3.8)
- $\mathfrak{p}_s := \{x \in \mathfrak{g},\ \mathrm{Ad}(e^{ts})x$ is bounded when $t \to \infty\}$ is the Lie algebra of P_s
- $\mathfrak{l}_s := \{x \in \mathfrak{g},\ [x, s] = 0\}$ is the Lie algebra of L_s,
- $\mathbb{V}_s := \{v \in \mathbb{V},\ \rho(e^{ts}v)$ is bounded when $t \to \infty\}$,
- $\mathbb{V}^0_s := \{v \in \mathbb{V},\ \rho(e^{ts}v) = v$ for any $t\}$,
- $\chi_s :=$ the dual of s respect to B, meaning that B induces an isomorphism $B : \mathfrak{h} \to \mathfrak{h}^*$ and $\chi_s := B(s)|_{\mathfrak{p}_s}$. A character of $\mathfrak{p}_s$ is a complex linear map $\mathfrak{p}_s \to \mathbb{C}$ which factors through the quotient $\mathfrak{p}_s/[\mathfrak{p}_s, \mathfrak{p}_s]$. Since $B(s, [\mathfrak{p}_s, \mathfrak{p}_s]) = 0$, then χ_s is a strictly antidominant character of $\mathfrak{p}_s$.

Remark 2.3.1 We have seen that given $s \in \sqrt{-1}\mathfrak{h}$, then χ_s is a strictly antidominant character of $\mathfrak{p}_s$. Conversely, given an antidominant character χ of $\mathfrak{p}_s$, then $\chi \in (\mathfrak{p}_s/[\mathfrak{p}_s, \mathfrak{p}_s])^* \cong (\mathfrak{z}_{L_s})^*$, where $\mathfrak{z}_{L_s}$ is the center of the Levi subalgebra $\mathfrak{l}_s$. Let $s_\chi = B^{-1}(\chi)$, then $s_\chi \in \mathfrak{z}_{L_s} \subset \sqrt{-1}\mathfrak{h}$.

Let (E, φ) be a Higgs pair, $s \in \sqrt{-1}\mathfrak{h}$ and σ be a reduction of structure group of E from G to P_s. Now, we recall two equivalent definitions of the degree of E

associated to σ and s. Let $s \in \sqrt{-1}\mathfrak{h}$ and χ_s be the strictly antidominant character of $\mathfrak{p}_s$, given by the dual of s with respect to B. By [39, Lemma 2.4] there is a natural number n such that $(\chi_s)^n$, that is antidominant character of $\mathfrak{p}_s$, lifts to a character $\hat{\chi}_s$ of P_s. Then $E_{P_s}(\hat{\chi}_s)$ is a line bundle and we define the **degree of E associated to σ and s** as follows

$$\deg E(\sigma, s) := \frac{1}{n} \deg E_{P_s}(\hat{\chi}_s) . \tag{2.3.5}$$

Equivalently, we can define $\deg E(\sigma, s)$ in terms of filtrations in the following way.

Proposition 2.3.1 ([39], Lemma 2.12) Let $\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$ be a representation coming from a representation $\rho : H \longrightarrow \mathrm{U}(\mathbb{V})$. Let $\pi : \mathfrak{g} \to \mathrm{Ker}(\rho)^{\perp}$ be the orthogonal projection with respect to B. Let $s \in \sqrt{-1}\mathfrak{h}$ and let $\lambda_1 < \ldots < \lambda_k$ be the real eigenvalues of $\rho(s)$ (this matrix is diagonalizable because it is unitary). Let $\mathbb{V}_j$ be the eigenspace associated to λ_j and define $\mathbb{V}_{\leq j} = \oplus_{j \leq i} \mathbb{V}_j$. If χ is an antidominant character of $\mathfrak{p}_A$ such that $s = s_\chi$, then it is an antidominant character of

$$P_{\mathbb{V},s} := \{g \in G, \rho(g)(\mathbb{V}_{\leq j}) \subset \mathbb{V}_{\leq j}\} = P_{\pi(s)} .$$

Let $\rho_a : G \to \mathrm{GL}(\mathbb{V})$ a representation such that $B(u, v) = \mathrm{Tr}(\rho_a^*(u)\rho_a(v))$ for every $u, v \in (\mathrm{Ker}\rho)^{\perp}$. (It exists, because all bi-invariant metrics are of this form). Then, for every reduction σ of the structure group of E to a parabolic subgroup $P_{\pi(s)}$, one has

$$\deg E(s, \sigma) = \lambda_k \deg E(\mathbb{V}) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg E(\mathbb{V}_{\leq i}) .$$

The adjoint bundle $\mathrm{Ad}(E)$ admits a real structure $\sigma_{\mathrm{Ad}(E)}$ (Example 1.7.2). The main change in the definition of stability for real (quaternionic) Higgs pairs with respect to the definition of stability of the underlying Higgs pairs is that we must consider only reductions preserved by $\sigma_{\mathrm{Ad}(E)}$. More specifically.

Definition 2.3.2 Let $\alpha \in \mathfrak{z}(\mathfrak{h})$. A $(\sigma_L, \pm)$-real (quaterionic) Higgs pair (E, φ, σ_E) is:

- α-**semistable** if for every $s \in \sqrt{-1}\mathfrak{h}$ and for every reduction σ of structure group of E to P_s such that

$$\sigma_{\mathrm{Ad}(E)} \mathrm{Ad}(E_{P_s}) = \mathrm{Ad}(E_{P_s})$$

and $\varphi \in H^0(X, E_{P_s}(\mathbb{V}_s) \otimes L)$, then

$$\deg E(s,\sigma) - B(s,\alpha) \geq 0\,,$$

- α-**stable** if for every $s \in \sqrt{-1}\mathfrak{h} \backslash \mathrm{Ker}(d\rho)$ and for every reduction σ of structure group of E to P_s such that

$$\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}(E_{P_s}) = \mathrm{Ad}(E_{P_s})$$

and $\varphi \in H^0(X, E_{P_s}(\mathbb{V}_s) \otimes L)$, then

$$\deg E(s,\sigma) - B(s,\alpha) > 0\,,$$

- α-**polystable** if it is α-semistable and furthermore, if for every $s \in \sqrt{-1}\mathfrak{h}$ and for every reduction σ of structure group of E to P_s such that

$$\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_{P_s} = \mathrm{Ad}E_{P_s},$$

$\varphi \in H^0(X, E_{P_s}(\mathbb{V}_s) \otimes L)$ and

$$\deg E(s,\sigma) - B(s,\alpha) = 0\,,$$

then there is a holomorphic reduction σ' of structure group of E_P to L_s such that

$$\sigma_{\mathrm{Ad}(E)}(\mathrm{Ad}(E_{L_s})) = \mathrm{Ad}(E_{L_s})$$

and $\varphi \in H^0(X, E_{L_s}(\mathbb{V}_s^0) \otimes L)$.

Remark 2.3.2 Let (E, φ, σ_E) be a $(c, \sigma_L, \pm)$-real (quaterionic) Higgs pair. If the underlying Higgs pair (E, φ) is α-(semi)stable or α-polystable, then (E, φ, σ_E) is α-semistable or α-polystable, respectively.

2.4 Moment map and its integral for Higgs pairs

This section follows [62, Section 2.3.3]. Let (M, ω) be a Kähler manifold with a hamiltonian action of a compact group $\mathcal{H}$ and let $\mu : M \to \mathrm{Lie}(\mathcal{H})$ be its moment map. It is well defined up to central elements. Let $c_{\mathcal{H}}$ be a central element in $\mathrm{Lie}(\mathcal{H})$, we define $\mu^{c_{\mathcal{H}}} := \mu - c_{\mathcal{H}}$.

Definition 2.4.1 Let $\langle , \rangle$ be a bilinear form of $\mathrm{Lie}(\mathcal{H})$. We consider

$$\lambda_t^{c_{\mathcal{H}}}(m;s) := \left\langle \mu^{c_{\mathcal{H}}}(e^{\sqrt{-1}ts}m), s \right\rangle \quad \text{, for every } t \in \mathbb{R}, m \in M \text{ and } s \in \mathrm{Lie}(\mathcal{H})\,.$$

The **maximal weight** $\lambda^{c_{\mathcal{H}}}(m,s)$ of the action of $s \in \mathrm{Lie}(\mathcal{H})$ on $m \in M$ is the real number

$$\lambda^{c_{\mathcal{H}}}(m;s) := \lim_{t\to\infty} \lambda_t^{c_{\mathcal{H}}}(m;s)\,.$$

Definition 2.4.2 The **integral of the moment map** $\mu^{c_{\mathcal{H}}}$ is the function

$$\begin{array}{cccc} \Psi^{c_{\mathcal{H}}}: & M \times \mathcal{H}^{\mathbb{C}} & \longrightarrow & \mathbb{R} \\ & (m, e^{\sqrt{-1}s}) & \longmapsto & \int_0^1 \lambda_t^{c_{\mathcal{H}}}(m;s)dt\,. \end{array} \tag{2.4.6}$$

Remark 2.4.1 ([62] Section 2.2) Fix $m \in M$. The integral of the moment map $\Psi^{c_{\mathcal{H}}}$ satisfies that the critical points of $\Psi^{c_{\mathcal{H}}}|_{m\times\mathcal{H}^{\mathbb{C}}}$ are the points of $\mathcal{H}^{\mathbb{C}}$ such that the moment map $\mu^{c_{\mathcal{H}}}$ annihilates and the restrictions $\Psi^{c_{\mathcal{H}}}|_{\{m\times e^{ts},\, t\in\mathbb{R}\}}$ are convex.

Let $\alpha \in \mathfrak{z}(\mathfrak{h})$. We also denote by α to the element in $\mathrm{Lie}(\mathcal{H})$ whose fibers are equal to α. Let $\langle , \rangle$ be the bilinear form of on $\mathrm{Lie}(\mathcal{H})$ induced by B. Let $\rho : G \to \mathrm{GL}(\mathbb{V})$ a representation coming from a unitary representation $\rho : H \to \mathrm{U}(\mathbb{V})$. Let X be a Riemann compact surface and ω be a σ_X-real Kähler form on X. Let L be a line bundle over X. Let E be a G-principal bundle over X and σ be a reduction of the structure group of E from G to H. Let $\mathbb{E}_H$ be the C^∞-bundle underlying to the reduced bundle E_H.

Example 2.4.1 Let $\mathcal{A}^{1,1}$ be the set of connections A on $\mathbb{E}_H$ such that $F_A^{0,2} = 0$. One has that $\mathcal{A}^{1,1}$ is a Kähler manifold with a hamiltonian action of the gauge group $\mathcal{H} = \Gamma(\mathbb{E}_H(H))$. The moment map is

$$\mu^\alpha(A) := \Lambda F_A - \sqrt{-1}\,\alpha\,, \tag{2.4.7}$$

where Λ denotes the contraction with ω. The maximal weight of the action of $s \in \mathrm{Lie}(\mathcal{H})$ on $A \in \mathcal{A}^{1,1}$ is given by

$$\lambda^\alpha(A,s) = \left\langle \mu^\alpha(e^{\sqrt{-1}s}A)\,,\, s \right\rangle\,.$$

Example 2.4.2 Let $\mathcal{S}$ be the space of smooth sections of $\mathbb{E}_H(\mathbb{V}) \otimes L$. $\mathcal{S}$ is a Kähler manifold with a hamiltonian action of the gauge group $\mathcal{H} = \Gamma(\mathbb{E}_H(H))$. The moment map is

$$\mu^\alpha(\varphi) := \mu_h(\varphi) - \sqrt{-1}\,\alpha\,, \tag{2.4.8}$$

where $\mu_h(\phi)$ is defined as in Equation (2.2.3). The maximal weight of the action of $s \in \mathrm{Lie}(\mathcal{H})$ on $\varphi \in \mathcal{S}$ is

$$\lambda^\alpha(\varphi, s) = \left\langle \mu^\alpha(\, e^{\sqrt{-1}\,s}\phi)\,,\, s \right\rangle\,.$$

Example 2.4.3 The product space $M := \mathcal{A}^{1,1} \times \mathcal{S}$ is a Kähler manifold with a hamiltonian action of the gauge group $\mathcal{H} = \Gamma(\mathbb{E}_H(H))$. The moment map is

$$\mu^\alpha(A, \varphi) := \Lambda F_A + \mu_h(\varphi) - \sqrt{-1}\,\alpha\,, \tag{2.4.9}$$

for every $(A, \phi) \in M$, where $\mu_h(\phi)$ is defined as in Equation (2.2.3) and Λ denotes the contraction with ω.

The maximal weight of the action of $s \in \mathrm{Lie}(\mathcal{H})$ on $m \in M$ is

$$\lambda^\alpha((A, \varphi), s) = \left\langle \mu^\alpha(e^{\sqrt{-1}\,s}A\,,\, e^{\sqrt{-1}\,s}\phi)\,,\, s \right\rangle\,.$$

Definition 2.4.3 The **section** $\epsilon_{h,\sigma,s} \in \Omega^0(E(\sqrt{-1}\mathfrak{h}))$ **associated to an element** $s \in \sqrt{-1}\,\mathfrak{h}$**, a reduction of structure group** h **of** E **from** G **to** H **and a reduction of structure group** σ **of** E **from** G **to** P_s is defined as follows: let E_h be the reduced H-bundle induced by h. The reduction σ defines a map:

$$\psi : E_h \to G/P_s. \tag{2.4.10}$$

If $e \in E_h$, then $\psi(e)$ is a parabolic subgroup of G conjugate to P_s. Given a parabolic subgroup P conjugate to P_s, and an strictly antidominant character χ of $\mathfrak{p}_s$, From [39, Lemma 2.6], there exist $s_{P,\chi} \in \sqrt{-1}\mathfrak{h}$ such that $P = P_{s_{P,\chi}}$. The map

$$\begin{array}{rccc} \xi : & E_h & \longrightarrow & \sqrt{-1}\mathfrak{h} \\ & e & \longmapsto & s_{\psi(e),\chi_s} \end{array} \tag{2.4.11}$$

induces a section $\epsilon_{h,\sigma,s}$ of $E_h(\sqrt{-1}\mathfrak{h})$.

Proposition 2.4.1 ([39], Theorem 2.24) Let (E, φ) be a Higgs pair, let σ be a reduction of structure group of E from G to H and $s \in \sqrt{-1}\mathfrak{h}$. Let A be the connection on $\mathcal{A}^{1,1}$ that corresponds to the holomorphic structure on E. The maximal weight of the action of s on $(A, \varphi) \in \mathcal{A}^{1,1} \times \mathcal{S}$ and the degree are related by

$$\lambda^{\alpha}(\,(A, \varphi)\,, \sqrt{-1}\epsilon_{h,\sigma,s}\,) = \deg E(s, \sigma) + B(\alpha, s)\,. \tag{2.4.12}$$

2.5 Hitchin-Kobayashi correspondence

We follow the approach in [39, Sections 2.9, 2.10, 2.11] checking that every construction is compatible with real structures. The main result of the chapter, that we prove in the subsequent sections, is the following.

Theorem 2.5.1 (Hitchin-Kobayashi correspondence) A $(c, \sigma_L, \pm)$-real (quaternionic) Higgs pair (E, φ, σ_E) is α-polystable if and only if there exists a σ_E-compatible Hermite-Einstein-Higgs reduction of structure group of E from G to H.

Corollary 2.5.2 A $(c, \sigma_L, \pm)$-real (quaternionic) Higgs pair (E, φ, σ_E) is α-polystable if and only if the underlying Higgs pair (E, φ) is α-polystable.

Proof: If a Higgs pair (E, φ) is α-polystable then (E, φ, σ_E) is α-polystable for any $(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)$-real (quaternionic) structure σ_E. (See Remark 2.3.2).

Conversely, in view of Theorem 2.5.1, it follows that α-polystability of a $(c, \sigma_L, \pm)$-real (quaternionic) Higgs pair (E, φ, σ_E) is equivalent to a σ_E-compatible solution of the Hermite-Einstein-Higgs equation (2.2.4). This implies polystability of the underlying Higgs pair by the Hitchin-Kobayashi correspondence [39, Theorem 2.24]. $\square$

2.6 Real infinitesimal automorphism space

First, we recall two definitions given in [39, Sections 2.9].

Definition 2.6.1 The **infinitesimal automorphism space** of a Higgs pair (E,φ) is the subset of sections

$$\mathrm{aut}(E,\varphi) := \{s \in H^0(E(\mathfrak{g})),\ \ d\rho(s)(\varphi) = 0\}\,.$$

Definition 2.6.2 The **semisimple infinitesimal automorphism space** $\mathrm{aut}^{ss}(E,\varphi))$ of (E,φ) as the subset of $\mathrm{aut}^{ss}(E,\varphi)$ given by

$$\mathrm{aut}^{ss}(E,\varphi) := \{s \in \mathrm{aut}(E,\varphi),\ \ s(x) \text{ is semisimple for any } x \in X\}\,.$$

Definition 2.6.3 Let $\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$ be a $(\sigma_G,\sigma_{\mathbb{V}})$-real (quaternionic) representation coming from $\rho : H \longrightarrow \mathrm{U}(\mathbb{V})$. Let V be the associated bundle to E via ρ. Let $g \in \Omega^0(E(\mathfrak{g}))$ and suppose that $\rho(g)$ has constant eigenvalues $\lambda_1 < \ldots < \lambda_r$, **the filtration** $V(g)$ **associated to** g is defined as follows: let V^i be the eigenbundle of eigenvalue λ_i and $V_{\leq i} := \oplus_{j=1}^{i} V^j$, for $1 \leq i \leq r$, then

$$V(g) := V_{\leq 1} \subset V_{\leq 2} \subset \ldots V_{\leq r}\,.$$

Let (E,φ,σ_E) be a $(\sigma_L,\pm)$-real (quaternionic) Higgs pair. The space $\mathrm{aut}(E,\varphi)$ admits the involution $\sigma_{\mathrm{aut}}(s)(x) := \sigma_{\mathrm{ad}(E)}(s(\sigma_X(x)))$, for every $s \in \mathrm{aut}(E,\varphi)$ and $x \in X$, where $\sigma_{\mathrm{ad}(E)}$ is the antiholomorphic involution of the adjoint bundle $\mathrm{ad}(E)$ defined on Example 1.7.3.

Definition 2.6.4 The **infinitesimal automorphisms** $\mathrm{aut}(E,\varphi,\sigma_E)$ **of** (E,φ,σ_E) is the set $s \in \mathrm{aut}(E,\varphi)$ such that the filtrations $V(s)$ and $V(\sigma_{\mathrm{aut}}(s))$ define two reductions of structure groups E_P and $\sigma_E E_P$ satisfying $\mathrm{Ad}(E_P) = \mathrm{Ad}(\sigma_E E_P)$.

Definition 2.6.5 The semisimple infinitesimal automorphisms space $\mathrm{aut}^{ss}(E,\varphi,\sigma_E)$ of (E,φ,σ_E) is the intersection of $\mathrm{aut}^{ss}(E,\varphi)$ and $\mathrm{aut}(E,\varphi,\sigma_E)$.

We denote by $H^0(X,E(\mathfrak{z}))^{\sigma_E}$ the intersection of $H^0(X,E(\mathfrak{z}))$ and $\mathrm{aut}(E,\varphi,\sigma_E)$.

Proposition 2.6.1 Let (E,φ,σ_E) be an α-polystable $(c,\sigma_L,\pm)$-real (quaterionic) Higgs pair, then (E,φ,σ_E) is α-stable if and only if

$$\mathrm{aut}^{ss}(E,\varphi,\sigma_E) = H^0(E(\mathfrak{z}))^{\sigma_E}. \tag{2.6.13}$$

If (E,φ,σ_E) is α-stable then

$$\mathrm{aut}(E,\varphi,\sigma_E) = H^0(E(\mathfrak{z}))^{\sigma_E}\,. \tag{2.6.14}$$

Proof: Suppose that $\mathrm{aut}^{ss}(E,\varphi,\sigma_E) = H^0(E(\mathfrak{z}))^{\sigma_{H^0(E(\mathfrak{z}))}}$. If (E,φ,σ_E) is not α-stable, then there is a reduction σ of the structure group of E to P_s, with $P_s \neq G$ such that $\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_{P_s} = \mathrm{Ad}E_{P_s}$ and

$$\deg E(s,\sigma) - B(s,\alpha) = 0\,,$$

furthermore, there is a reduction E_{L_s} of the structure group of E to a Levi subgroup L_s such that

$$\sigma_{\mathrm{Ad}(E)}(\mathrm{Ad}E_{L_s}) = \mathrm{Ad}E_{L_s} \tag{2.6.15}$$

and $\varphi \in H^0(X, E_{L_s}(\mathbb{V}^0_s)\otimes L)$. Let $\psi_{\sigma,s} \in \mathrm{aut}^{ss}(E,\varphi)$ be the section that is equal to s in the fibres, it exist because the adjoint action of L_s in $\mathfrak{g}$ fixes s. (See [39, Proposition 2.14]). The section $\psi_{\sigma,s} \in \mathrm{aut}^{ss}(E,\varphi,\sigma_E)$ by (2.6.15). Then $\psi_{\sigma,s} \in H^0(E(\mathfrak{z}))^{\sigma_E}$ by hypothesis (2.6.13), but $s_{\sigma,\chi}$ is not central because $P_s \neq G$ and this is a contradiction.

Conversely, it suffices to prove (2.6.14). Let ϵ be an element in $\mathrm{aut}(E,\varphi,\sigma_E)$. The element $\epsilon \in H^0(X,E(\mathfrak{g}))$ corresponds to a equivariant map $\phi_\epsilon : E \to \mathfrak{g}$ and taking quotients by G, we obtain a constant map $\phi'_\epsilon : X \to \mathfrak{g} /\!\!/ G$. (See [39, Proposition 2.14]). Let y be the image of ϕ'_ϵ and $\pi : \mathfrak{g} \to \mathfrak{g} /\!\!/ G$ be the quotient map. Let Y be the space $\pi^{-1}(y)$ and $\mathcal{O} \subset Y$ the G-closed semisimple orbit. The spaces Y and $\mathcal{O}$ are invariant under $d\sigma_G$. The map sending an element y in Y to its semisimple part y_s in $\mathcal{O}$, induces a map $H^0(E(Y)) \to H^0(E(\mathcal{O}))$ that sends ϵ to ϵ_s. Let $\epsilon_n = \epsilon - \epsilon_s$ be the nilpotent part. Since σ_G preserves the semisimple and nilpotent parts, one can see that $\epsilon_n, \epsilon_s \in \mathrm{aut}(E,\varphi,\sigma_E)$.

Finally, we have to prove that $\epsilon_s \in H^0(E(\mathfrak{z}))^{\sigma_E}$ and $\epsilon_n = 0$. First, we prove that $\epsilon_s \in H^0(E(\mathfrak{z}))^{\sigma_E}$. In fibres, ϵ_s is equal to $y_s \in \mathcal{O}$. By [39, Lemma 2.15] there is an element $g \in G$ such that

$$\mathrm{Ad}(g^{-1})y_s = u = u_i + \sqrt{-1}u_r\,,$$

where u_i, u_r are two elements of $\mathfrak{h}$ satisfying:

(1) $[u_i,u_r] = 0\,,$

(2) there is an element $a \in \mathfrak{h}$ such that $\mathrm{Ker}(\mathrm{ad}(s)) = \mathrm{Ad}(g)\mathrm{Ker}(\mathrm{ad}(a))$.

Condition (2) implies that the torus $T_a := \{e^{\sqrt{-1}ta}, t \in \mathbb{R}\}$ is precisely the torus generated by $\{u_i, u_r\}$. Let $E_0 := \{e \in E, \phi_{\epsilon_s}(e) = u\}$ be the reduced bundle E_{G_0}, where $G_0 := \{g \in G, \mathrm{Ad}(g)u = u\}$. We have that G_0 is invariant under σ_G and the reduction is real $\sigma_E(E_{G_0}) = E_{G_0}$. The bundle E_0 induces reductions σ^+, σ^- to $P^+ := P_{\sqrt{-1}u_i}$ and $P^- := P_{-\sqrt{-1}u_i}$. These reductions corresponds to filtrations $V(\phi_{\sqrt{-1}u_i})$ and $V(\phi_{-\sqrt{-1}u_i})$, respectively, where $\phi_{\sqrt{-1}u_i}$ is the section that is equal to u_i in the fibres. Since $\phi_{\sqrt{-1}u_i} \in \mathrm{aut}(E, \varphi, \sigma_E)$, then

$$\sigma_{\mathrm{Ad}(E)}(\mathrm{Ad}E_{P^\pm}) = \mathrm{Ad}E_{P^\pm}\,.$$

Applying stability conditions for both subgroups P^+, P^- we obtain

$$\deg E(s, \sigma^+) - B(s, \alpha) \geq 0\,,\ \deg E(s, \sigma^-) - B(s, \alpha) \geq 0\,.$$

Since $\deg E(s, \sigma^+) - B(s, \alpha) = \deg E(s, \sigma^-) - B(s, \alpha)$, then $\deg E(s, \sigma) - B(s, \alpha) = 0$ and by α-stability of (E, φ, σ_E), ϵ_s is central.

Now, we prove that $\epsilon_n = 0$ by contradiction. There is a filtration

$$W := \dots W^{-k} \subset W^{-k+1} \subset \dots \subset W^{k-1} \subset W^k \subset \dots \subset E(\mathfrak{g})\,,$$

on $E(\mathfrak{g})$ defined by three conditions: $\mathrm{ad}(\epsilon_n)W^j \subset W^{j-2}$, $\mathrm{ad}(\epsilon_n)^{j+1}W^j = 0$, and the induced map on graded spaces $\mathrm{Gr}(\mathrm{ad}\epsilon_n) : \mathrm{Gr}(W^j) \to \mathrm{Gr}(W^{-j})$ is an isomorphism. The filtration W induces a reduction of the structure group of E from G to the Jacobson parabolic subgroup P. Since W is invariant under σ_{ad}, then $\sigma_{\mathrm{Ad}}\mathrm{Ad}(E_P) = \mathrm{Ad}(E_P)$. If we choose an antidominant character χ perpendicular to $\mathfrak{z}$ then

$$\deg E(\sigma, s_\chi) + B(s_\chi, \alpha) = \sum_{j \in \mathbb{Z}} j \deg(W^j) \leq 0$$

and we obtain a contradiction with stability. $\square$

Definition 2.6.6 A Higgs pair (E, φ) is **simple** if $\mathrm{Aut}(E, \varphi) = Z(G) \cap \ker(\rho)\,.$

Definition 2.6.7 A $(c, \sigma_L, \pm)$-real (quaterionic) Higgs pair (E, φ, σ_E) is **simple** if $\mathrm{Aut}(E, \varphi, \sigma_E) = Z(G)^{\sigma_G} \cap \ker(\rho)\,.$

2.7 Real Jordan-Hölder reduction

Let (E, σ_E) be a (σ_X, σ_G, c)-real G-bundle over (X, σ_X). The following lemma is a consequence of [39, Lemma 2.16].

Lemma 2.7.1 Let G' be a subgroup of G such that the normalizer $N_G(\mathfrak{g})$ of $\mathfrak{g}$ in G is equal to $\mathrm{Lie}(G')$. Reductions of E from G to G' satisfying

$$\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_{G'} = \mathrm{Ad}E_{G'}\,, \tag{2.7.16}$$

are in one-to-one correspondence with subbundles $F \subset E(\mathfrak{g})$ of Lie algebras satisfying:

(1) $\sigma_{\mathrm{ad}(E)}(F) = F$,

(2) for any $x \in X$ and any trivialization $E_x \simeq G$, the fibre F_x is conjugate with $\mathrm{Lie}(G')$, via the induced trivialization $E(\mathfrak{g})_x \simeq \mathfrak{g}$.

Let $P \subset G$ be a parabolic subgroup. Let E_P be a reduction such that

$$\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_P = \mathrm{Ad}E_P\,. \tag{2.7.17}$$

Let $L \subset P$ be a Levi subgroup of P. Reduction E_L of the structure group of E_P from P to L such that

$$\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}(E_L) = \mathrm{Ad}(E_L)\,. \tag{2.7.18}$$

Conditions (2.7.17) and (2.7.18) imply that $\sigma_{\mathrm{ad}(E)} = d(\sigma_{\mathrm{Ad}(E)})$ preserves $E_P(\mathfrak{u})$, $E_P(\mathfrak{p})$ and $E_P(\mathfrak{l})$, so we have real structures $\sigma_{E_P(\mathfrak{u})}, \sigma_{E_P(\mathfrak{p})}$ and $\sigma_{E_P(\mathfrak{l})}$ on $E_P(\mathfrak{u})$, $E_P(\mathfrak{p})$ and $E_P(\mathfrak{l})$, respectively and there is an exact sequence of real vector bundles

$$0 \longrightarrow (E_P(\mathfrak{u}), \sigma_{E_P(\mathfrak{u})}) \longrightarrow (E_P(\mathfrak{p}), \sigma_{E_P(\mathfrak{p})}) \longrightarrow (E_P(\mathfrak{l}), \sigma_{E_P(\mathfrak{l})}) \longrightarrow 0. \tag{2.7.19}$$

Lemma 2.7.2 Let E_P be a reduction satisfying (2.7.17). Reductions E_L of the structure group of E_P from from P to L satisfying (2.7.18) are in one-to-one correspondence with splittings of (2.7.19) that are $(\sigma_{E_P(\mathfrak{l})}, \sigma_{E_P(\mathfrak{p})})$-real.

Proof: We follow [39, Lemma 2.17]. Taking $G' = L$, by Lemma 2.7.1, the space of reductions of E from G to L satisfying (2.7.17) and (2.7.18) is in one-to-one correspondence with subbundles $F \subset E_P(\mathfrak{p})$ such that $\sigma_{\mathrm{ad}(E)}(F) = F$ and F_x is conjugate to $\mathfrak{l}$ once we identify $E(\mathfrak{g})_x$ with $\mathfrak{g}$ for any $x \in X$. But this set is one-to-one correspondence with $(\sigma_{E_P(\mathfrak{l})}, \sigma_{E_P(\mathfrak{p})})$-real splittings. $\square$

Let (E, φ, σ_E) an α-polystable $(c, \sigma_L, \pm)$-real (quaternionic) Higgs pair, such that is not α-stable, then from Proposition 2.6.1, there is a section η of the bundle $E([\mathfrak{g}, \mathfrak{g}])$. Taking $\eta + \sigma_{\mathrm{aut}}\eta$ if neccesary, we can suppose that η is invariant under σ_{ad}. Following the last part of Proposition 2.6.1, the image of η is contained in an adjoint orbit in $\mathfrak{g}$ which contains the element $u = u_r + iu_i$ such that u_i, u_r are commuting elements of $\mathfrak{h}$ and there is an element $a \in \mathfrak{h}$ such that T_a is equal to the torus generated by u_r, u_i. The elements u and a satisfy

$$d\sigma_G u = u\,,\; d\sigma_G a = a\,.$$

The subgroup $H_1 = Z_H(a)$ is preserved by σ_G. On the other hand, by Lemma 1.7.1, η gives a real equivariant map ϕ_s from E to $\mathfrak{g}$, so

$$E_1 := \{e \in E, \phi_s(e) = u\}$$

preserves σ_E, therefore E_1 is a $(\sigma_X, \sigma_G|_{H_1^{\mathbb{C}}}, c)$-real $H_1^{\mathbb{C}}$-bundle. Since ρ is a real (quaternionic) representation, the vector space $\mathbb{V}_1 := \{v \in \mathbb{V}, \rho(a)v = 0\}$ has a real (quaternionic) structure $\sigma_{\mathbb{V}_1} = \sigma_{\mathbb{V}}|_{\mathbb{V}_1}$ and $\rho|_{H_1}$ is a $(\sigma_G, \mathbb{V}_1)$-real (quaternionic) representation.

Proposition 2.7.1 *The $(c, \sigma_L, \pm)$-real (quaternionic) Higgs pair $(E_1, \sigma_E|_{E_1}, \phi_1 := \phi|_{E_1})$ is α-polystable.*

Proof: First, we prove that is α-semistable. Let $s \in \sqrt{-1}\,\mathfrak{h}_1 = \mathrm{Lie}(H_1)$. We consider the parabolic subgroup

$$P_{1,s} := \{g \in H_1\,,\; e^{ts}ge^{-ts} \text{ is bounded when } t \to \infty\}\,.$$

Any reduction σ_1 to $P_{1,s}$ such that $\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_{P_{s,1}} = \mathrm{Ad}E_{P_{s,1}}$ can be extended to a real reduction σ to P_s such that $\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_{P_s} = \mathrm{Ad}E_{P_s}$. Moreover,

$$\deg E(\sigma, s) = \deg E_1(\sigma_1, s)\,.$$

Therefore, since (E, φ, σ_E) is α-semistable then $(E_1, \phi_1 := \phi|_{E_1}, \sigma_E|_{E_1})$ is also α-semistable. From Proposition 2.7.2, it follows that to prove α-polystability of $(E_1, \phi_1, \sigma_E|_{E_1})$ it is enough to see that if $s \in \sqrt{-1}\mathfrak{h}$ and σ is a reduction to P_s, satisfying

$\sigma_{\mathrm{Ad}(E_1)}\mathrm{Ad}E_{P_{1,s}} = \mathrm{Ad}E_{P_{1,s}}$ and

$$\deg E_1(\sigma_1, s) - B(s, \alpha) = 0\,,$$

then there is a real splitting ω_1 of the following exact sequence of real (quaternionic) vector bundles

$$0 \longrightarrow (E_{1\sigma_1}(\mathfrak{u}_1), \sigma_{E_{1\sigma_1}(\mathfrak{u}_1)}) \longrightarrow (E_{1\sigma_1}(\mathfrak{p}_1), \sigma_{E_{1\sigma_1}(\mathfrak{p}_1)}) \longrightarrow (E_{1\sigma_1}(\mathfrak{l}_1), \sigma_{E_{1\sigma_1}(\mathfrak{l}_1)}) \longrightarrow 0\,.$$

This sequence fits, taking η trivial, in the last row of the following diagram of real (quaternionic) vector bundles

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (E_\sigma(\mathfrak{u}), \sigma_{E_\sigma(\mathfrak{u})}) & \longrightarrow & (E_\sigma(\mathfrak{p}), \sigma_{E_\sigma(\mathfrak{p})}) & \longrightarrow & (E_\sigma(\mathfrak{l}), \sigma_{E_\sigma(\mathfrak{l})}) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & (E_{1\sigma_1}(\mathfrak{u}), \sigma_{E_{1\sigma_1}(\mathfrak{u})}) & \longrightarrow & (E_{1\sigma_1}(\mathfrak{p}), \sigma_{(E_1)_{\sigma_1}(\mathfrak{p})}) & \longrightarrow & (E_{1\sigma_1}(\mathfrak{l}), \sigma_{E_{1\sigma_1}(\mathfrak{l})}) & \longrightarrow & 0 \\
 & & \downarrow= & & \downarrow= & & \downarrow= & & \\
0 & \to & \bigoplus_{\eta\in T^\vee}(E_{1\sigma_1}(\mathfrak{u}_\eta), \sigma_{E_{1\sigma_1}(\mathfrak{u}_\eta)}) & \to & \bigoplus_{\eta\in T^\vee}(E_{1\sigma_1}(\mathfrak{p}_\eta), \sigma_{E_{1\sigma_1}(\mathfrak{p}_\eta)}) & \to & \bigoplus_{\eta\in T^\vee}(E_{1\sigma_1}(\mathfrak{l}_\eta), \sigma_{E_{1\sigma_1}(\mathfrak{l}_\eta)}) & \to & 0
\end{array}$$

where $T^\vee = \mathrm{Hom}(T, S^1)$ is the group of characters of T and

$$\mathfrak{u} = \bigoplus_{\eta\in T^\vee} \mathfrak{u}_\eta, \quad \mathfrak{p} = \bigoplus_{\eta\in T^\vee} \mathfrak{p}_\eta, \quad \mathfrak{l} = \bigoplus_{\eta\in T^\vee} \mathfrak{l}_\eta.$$

Therefore, there is a real splitting ω_1 if and only if there is a real splitting ω in the sequence

$$0 \longrightarrow (E_\sigma(\mathfrak{u}), \sigma_{E_\sigma(\mathfrak{u})}) \longrightarrow (E_\sigma(\mathfrak{p}), \sigma_{E_\sigma(\mathfrak{p})}) \longrightarrow (E_\sigma(\mathfrak{l}), \sigma_{E_\sigma(\mathfrak{l})}) \longrightarrow 0\,.$$

But this is true by Lemma 2.7.2, because (E, φ, σ_E) is α-polystable.

On the other hand, the real section $\varphi_1 \in H^0(E(\mathbb{V}) \otimes L)$ if and only if

$$\rho(\omega_1 s_{\sigma,\chi}(\varphi)) = 0\,. \tag{2.7.20}$$

Since (E, φ) is α-polystable, then $\rho(\omega s_{\sigma,\chi}(\varphi)) = 0$ and this implies (2.7.20).

$\square$

We can iterate this process and at the end, we obtain a real Higgs pair that is α-stable and that is called **a real Jordan-Hölder reduction**.

Remark 2.7.1 the uniqueness (up to conjugations) comes from the uniqueness in the non real case.

2.8 Stability implies existence of solutions

2.8.1 Real simplified Higgs pairs

Let X be a compact Riemann surface, G be a complex reductive Lie group, H be a maximal compact Lie subgroup of G and $B : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ be a a non degenarate bi-invariant metric on $\mathfrak{h} = \mathrm{Lie}(H)$. Let $\mathbb{V}$ a complex vector space equipped with a Hermitian metric $\langle \cdot, \cdot \rangle$. Let $\rho : G \to \mathrm{GL}(\mathbb{V})$ be a representation. By abuse of notation, we denote also by ρ the restriction $\rho : H \to \mathrm{U}(\mathbb{V})$.

Definition 2.8.1 A **simplified Higgs pair** is a pair (A, ϕ), where A is a connection on a C^∞ H-bundle $\mathbb{E}_H$ over X and ϕ is a holomorphic section of $V := \mathbb{E}_H(\mathbb{V})$, meaning that $\bar{\partial}_A(\phi) := A^{0,1}(\phi) = 0$.

Remark 2.8.1 There is a bijective correspondence between connections on $\mathbb{E}_H$ and complex structures on the G-bundle $\mathbb{E} := \mathbb{E}_H \times_H G$. (See [61, Lemma 2.1]). As a consequence, Higgs pairs can be considered as simplified Higgs pairs taking as a group $G \times \mathbb{C}^*$ and considering $\phi = \varphi \in E^L(\mathbb{V})$, where E^L is defined on Equation (2.1.2). We have to fix a flat connection on L. (See [25]).

Remark 2.8.2 By consistency with the rest of the Chapter, we consider X a Riemann surface, but all the results can be generalized to Kähler manifolds as in [61].

Let σ_X be a real structure on X, σ_G be a real structure on G. Changing H if neccesary, we can suppose by Corollary 1.3.5 that $\sigma_G(H) = H$. Let $\sigma_{\mathbb{V}}$ be a real (quaternionic) structure on $\mathbb{V}$ such that is compatible with the product $\langle \cdot, \cdot \rangle$ and

$\rho : H^{\mathbb{C}} \longrightarrow \mathrm{GL}(\mathbb{V})$ is a $(\sigma_G, \sigma_{\mathbb{V}})$-real (quaternionic) representation. Let $Z_2^{\sigma_G}$ be the subgroup of elements of the center Z of G of order two that are invariants under σ_G.

Definition 2.8.2 A $(\sigma_X, \sigma_G, c, \sigma_V, \pm)$-**real (quaternionic) structure** on a simplified Higgs pair (A, ϕ) is a (σ_X, σ_G, c)-real structure on $\sigma_{\mathbb{E}}$ a C^∞-bundle $\mathbb{E}$ such that

- A is a $\sigma_{\mathbb{E}}$-compatible connection on $\mathbb{E}$
- $\sigma_V(\phi) = \pm\sigma_X^*(\phi)$,
  with σ_V being the involution on $V = \mathbb{E}(\mathbb{V})$ induced by $\sigma_{\mathbb{E}}$ and $\sigma_{\mathbb{V}}$.

The triple $(A, \phi, \sigma_{\mathbb{E}})$ will be referred as a $(\sigma_X, \sigma_G, c, \sigma_V, \pm)$-**real (quaternionic) simplified Higgs pair**.

Remark 2.8.3 Let (E, σ_E) be a (σ_X, σ_G, c)-real (quaterionic) G-bundle. In view of Proposition 1.6.4, there is bijective correspondence between the space of (σ_X, σ_E)-compatible connections on E_H and σ_E-real complex structures on the underlying smooth G-bundle $\mathbb{E}$. As a consequence, real (quaternionic) Higgs pairs can be considered as simplified Higgs pairs taking as a group $G \times \mathbb{C}^*$, equipped with the involution $\sigma_G \times \mathrm{conj}$, and $\phi = \varphi \in E^L(\mathbb{V})$, fixing a flat connection on L. (See [25])

Let (A, ϕ) a simplified Higgs pair, $s \in \sqrt{-1}\mathfrak{h}$ and σ be a reduction of structure group of $\mathbb{E}$ from G to P_s. There are two equivalent definitions of $\deg \mathbb{E}(\sigma, s)$. One is given in (2.3.5) and the other in Proposition 2.3.1. We define the stability for real (quaternionic) simplified Higgs pair following the general definition of [25], that in this particular case, coincide with the one of [8]. The main change is that we consider only reductions preserved by $\sigma_{\mathrm{Ad}(E)}$, the antiholomorphic involution defined in Example 1.7.2.

Definition 2.8.3 Let $\alpha \in \mathfrak{z}(\mathfrak{h})$. A $(\sigma_X, \sigma_G, c, \sigma_V, \pm)$-real (quaternionic) simplified Higgs pair $(A, \phi, \sigma_{\mathbb{E}})$ is α-**stable** if for every $s \in \sqrt{-1}\mathfrak{h}$ and every reduction σ of the structure group of $\mathbb{E}$ from G to P_s such that

- $\phi \in \Gamma(\mathbb{E}(\mathbb{V})^-(\sigma, \chi_s))$, where $\mathbb{E}(\mathbb{V})^-(\sigma, \chi_s) \subset \mathbb{E}(\mathbb{V})$ is the subbundle where $g_{\sigma,s}(x)$ acts negatively for all $x \in X$, where $g_{\sigma,s} \in \Gamma(E(\sqrt{-1}\mathfrak{h}))$ is the section whose fibres are equal to s.

- $\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}\mathbb{E}_{P_s} = \mathrm{Ad}\mathbb{E}_{P_s}$,

then

$$\deg \mathbb{E}(\sigma, s) - \int_X B(s, \alpha) > 0\,.$$

Remark 2.8.4 A $(c, \sigma_L, \pm)$-real (quaterionic) Higgs pair (E, φ, σ_E) is α-stable if and only if the correspondent real (quaternionic) simplified pair $(A, \phi, \sigma_{\mathbb{E}})$ is α-stable.

2.8.2 Moment map for simplified Higgs pairs

In this subsection we apply the definitions in Section 2.4 to the simplified Higgs pairs case. Let $\mathbb{E}$ be a C^∞ G-bundle, h a reduction of the structure group from $\mathbb{E}_H$ be a C^∞ H-bundle. Let $\mathcal{A}^{1,1}$ be the set of connections A on $\mathbb{E}_H$ such that $F_A^{0,2} = 0$. Let $\rho : H \to \mathrm{U}(n)$ be a unitary representation. Let $\mathcal{S}$ be the space of sections of the associated vector bundle $\mathbb{E}_H(\mathbb{V})$ of $\mathbb{E}_H$ via ρ. The product space $M := \mathcal{A}^{1,1} \times \mathcal{S}$ is a Kähler manifold with a hamiltonian action of the gauge group $\mathcal{H} = \Gamma(\mathbb{E}_H(H))$. This is a particular case of Section 2.4. Let $\alpha \in \mathfrak{z}(\mathfrak{h})$. We also denote by α to the element in $\mathrm{Lie}(\mathcal{H})$ whose fibers are equal to α. Let $\langle , \rangle$ be the bilinear form of on $\mathrm{Lie}(\mathcal{H})$ induced by B. The moment map is

$$\mu^\alpha(A, \phi) := \Lambda F_A + \mu(\phi) - \sqrt{-1}\,\alpha\,, \tag{2.8.21}$$

for every $(A, \phi) \in M$, where $\mu(\phi)$ is defined as in Equation (2.2.3). The maximal weight of the action of $s \in \mathrm{Lie}(\mathcal{H})$ on $m \in M$ is

$$\lambda^\alpha((A, \phi), s) = \left\langle \mu^\alpha(e^{\sqrt{-1}\,s}A\,,\, e^{\sqrt{-1}\,s}\phi)\,,\, s \right\rangle\,.$$

The integral of the moment map is given by

$$\Psi^\alpha((A, \phi), e^{\sqrt{-1}s}) = \int_0^1 \left\langle \mu^\alpha(e^{\sqrt{-1}\,s}A\,,\, e^{\sqrt{-1}\,s}\phi)\,,\, s \right\rangle\,, \tag{2.8.22}$$

for every $(A, \phi) \in M$ and $e^{\sqrt{-1}s} \in \mathcal{H}$.

Remark 2.8.5 By Remark 2.4 if we fix a simplified Higgs pair (A, ϕ) then $e^{\sqrt{-1}s} \in \mathcal{H}$ is a critical point of Ψ^α if and only if $e^{\sqrt{-1}s}(A, \phi)$ is a solution of the moment map (2.8.21). In this case, the restrictions of the moment map in Remark 2.4 are always convex.

Proposition 2.8.1 Let $(A, \phi, \sigma_{\mathbb{E}})$ a $(\sigma_X, \sigma_G, c, \sigma_V, \pm)$-real (quaterionic) simplified Higgs pair, let σ be a reduction of the structure group and $s \in \sqrt{-1}\mathfrak{h}$, then

$$\deg \mathbb{E}(\sigma, s) - \int_X B(s, \alpha) = \lambda^{\alpha}((A, \phi), -\sqrt{-1}g_{\sigma,s}) ,$$

where $g_{\sigma,s} \in \Gamma(E(\sqrt{-1}\mathfrak{h}))$ is the section whose fibres are equal to s.

Proof: Is a consequence of [62, Lemma 2.3.3] and [62, Section 2.6.] $\square$

Remark 2.8.6 Let $\alpha \in \mathfrak{z}(\mathfrak{h})$. In view of the moment map interpretation of the degree of a real (quaterionic) Higgs pair (E, φ, σ_E) with respect a reduction σ of the structure group and $s \in \sqrt{-1}\mathfrak{h}$ (Proposition 2.4.1) and the moment map interpretation of the degree of a $(\sigma_X, \sigma_G, c, \sigma_V, \pm)$-real (quaterionic) simplified Higgs pair (A, φ) with respect to σ and s (Proposition 2.8.1), if (E, ϕ, σ_E) is α-stable then (A, φ) is α-stable.

2.8.3 Main theorem

Let $\mathbb{E}$ be a C^{∞} G-bundle and $\mathbb{E}_H$ a reduced bundle. We denote by $\mathcal{G}$ and $\mathcal{H}$ the gauge groups $\Gamma(\mathbb{E}(G))$ and $\Gamma(\mathbb{E}_H(H))$, respectively. In Example 1.7.3 we defined $\sigma_{\mathrm{ad}(E)}$. The real structures σ_X and $\sigma_{\mathrm{ad}(E)}$ induce an involution $\sigma_{\mathrm{Lie}(\mathcal{G})}$ on $\mathrm{Lie}(\mathcal{G})$.

Definition 2.8.4 A $(\sigma_X, \sigma_G, c, \sigma_V, \pm)$-real (quaternionic) simplified Higgs pair $(A, \phi, \sigma_{\mathbb{E}})$ is **simple** if there is no semisimple element $u \in \mathrm{Lie}(\mathcal{G})$ invariant under $\sigma_{\mathrm{Lie}(\mathcal{G})}$ such that

$$\left\langle \langle d\mu^{\alpha} \, , \, u \rangle \, , \, -\sqrt{-1}u \right\rangle (A, \phi) \neq 0 \, .$$

Theorem 2.8.5 (Main) Let $\alpha \in \mathfrak{z}(\mathfrak{h})$. Fix a simple and α-stable $(\sigma_X, \sigma_G, c, \sigma_V, \pm)$-real (quaternionic) simplified Higgs pair $(A, \phi, \sigma_{\mathbb{E}})$. There is an element $s \in \mathcal{G}/\mathcal{H}$ on which Ψ^{α} attains its minimum.

2.8.4 Proof of the main theorem

We fix a simple and α-stable $(\sigma_X, \sigma_G, c, \sigma_V, \pm)$-real (quaternionic) simplified Higgs pair $(A, \phi, \sigma_{\mathbb{E}})$. We denote by $\mathrm{Met}_2^p := L_2^p(E(\sqrt{-1}\,\mathfrak{h}))$ the completion of $\Omega^0(E(\sqrt{-1}\,\mathfrak{h}))$

with respect to the Sobolev norm

$$\|s\|_{L^p_2} = \|s\|_{L^p} + \|d_A s\|_{L^p} + \|\nabla d_A s\|_{L^p},$$

where $d_A s$ is the covariant derivative of s with respect to A and ∇ is the tensor product of the Levi-Civita connection and d_A. Via exponential map, $\Omega^0(E(\sqrt{-1}\,\mathfrak{h}))$ is isomorphic to $\mathcal{G}/\mathcal{H}$. Consider the bounded metric space

$$\mathrm{Met}^p_{2\,B} := \{s \in \mathrm{Met}^p_2, \text{ s.t. } \|\mu(e^s(A,\phi))\|^p_{L^p} \le B\,\}.$$

Proposition 2.8.2 If a metric minimize Ψ^α in $\mathrm{Met}^p_{2\,B}$, then it also minimize Ψ^α on Met^p_2.

Proof: Suppose that $s \in \mathrm{Met}^p_{2\,B}$ is a minimum of Ψ^α. Let $B = e^s(A), \Theta = e^s(\phi)$. We define the differential operator

$$L(u) := \sqrt{-1}\frac{\partial}{\partial t}\mu(e^{tu}(B,\Theta))|_{t=0}, \text{ for every } u \in L^p_2(E(\sqrt{-1}\mathfrak{h}))\,.$$

It suffices to prove that $\mathrm{Ker}(L) = 0$. Indeed, following the arguments in [62, 2.5.2.1], L is a Fredholm operator, it has index zero and modulo a compact operator is elliptic self-adjoint. If $\mathrm{Ker}(L) = 0$, then L is surjective and there is u such that $L(u) = \mu^\alpha(B,\theta)$. By [24, (Equation 3.28)] $\mu^\alpha(B,\theta) = 0$, and this implies that s minimize Ψ^α on Met^p_2. (See Remark 2.8.5).

We have to prove that $\mathrm{Ker}(L) = 0$. Let $0 \ne u \in \mathrm{Ker}(L)$. The element $\frac{u+\sigma_{\mathrm{Lie}(\mathcal{G})}(u)}{2} \in \mathrm{Ker}(L)$. Then $\left\langle -\sqrt{-1}L(\frac{u+\sigma_{\mathrm{Lie}(\mathcal{G})}(u)}{2})\,,\, \frac{u+\sigma_{\mathrm{Lie}(\mathcal{G})}(u)}{2}\right\rangle = 0$. By [62, Equation 2.13], $\frac{u+\sigma_{\mathrm{Lie}(\mathcal{G})}(u)}{2}$ leaves (B,θ) invariant, therefore (B,θ,σ_E) is not simple and (A,ϕ,σ_E) is not simple, and this is a contradiction.

□

Proposition 2.8.3 There exist positive constants C_1, C_2 such that

$$\sup|s| \le C_1\Psi^\alpha((A,\phi),e^s) + C_2, \text{ for any } s \in \mathrm{Met}^p_{2\,B}\,.$$

Proof: Repeating the same arguments that in [61, Section 6], it suffices to prove that

$$\|s\|_{L^1} \le C_1\Psi^\alpha((A,\phi),e^s) + C_2\,. \tag{2.8.23}$$

Suppose that C_1, C_2, satisfying (2.8.23) do not exist, then there is a sequence $\{u_i\} \in \mathrm{Met}^p_{2\,B}$ converging weakly to some $u_\infty \in \mathrm{Met}^p_{2\,B}$, such that

$$\lambda^\alpha((A,\phi),u_\infty) \leq 0\,. \tag{2.8.24}$$

Indeed, we follow [62, Lemma 2.5.4] if C_1, C_2 do not exist, then there is a sequence of real numbers $\{c_j\}$ and a sequence of $\{s_j\}$ of metrics in $\mathrm{Met}^p_2(B)$ such that $\lim_{j\to\infty} C_j = \infty$ and $\|s_j\|_{L^1} \geq C_j \Psi^\alpha(e^{s_j})$. Let $l_j := \|s_j\|_{L^1}$ and $u_j := \frac{s_j}{l_j}$, then $\|u_j\|_{L^1} = 1$ and $\sup |u_j| \leq C$.

Passing to a subsequence if necessary, we can suppose that $\{u_j\}$ converge to an u_∞ weakly on $L^2_1(E(i\mathfrak{h}))$ and $\lambda((A,\phi),u_\infty) \leq 0$. By definition of the integral of the moment map

$$\frac{1}{C_j} \geq \frac{\Psi^\alpha(e^{s_j})}{\|s_j\|} \geq \frac{l_j - t}{l_j}\lambda^\alpha_t((A,\phi),-\sqrt{-1}u_j) + \frac{1}{l_j}\int_0^t \lambda^\alpha_t((A,\phi),-\sqrt{-1}u_j)dt =$$
$$= \frac{l_j - t}{l_j}(\lambda^\alpha_t(A,-\sqrt{-1}u_j) + \lambda^\alpha_t(\phi,-\sqrt{-1}u_j)) + \frac{1}{l_j}\int_0^t (\lambda^\alpha_l(A,-\sqrt{-1}u_j) + \lambda^\alpha_l(\phi,-\sqrt{-1}u_j))dl.$$

The inequalities implies that $\|\bar\partial_A(u_j)\|$ is bounded so there is a subsequence $\{u_j\}$ converging to an u_∞. In addition,

$$\lambda^\alpha((A,\phi),u_\infty) \leq \lim_{t\to\infty} \lambda^\alpha_t((A,\phi),-\sqrt{-1}u_j) \leq 0\,.$$

The subsequence $\{\sigma_{\mathrm{ad}(E)}(u_i)\} \in \mathrm{Met}^p_{2\,B}$ converges weakly to $\sigma_{\mathrm{ad}(E)}u_\infty \in \mathrm{Met}^p_{2\,B}$, and

$$\lambda^\alpha((A,\phi),\sigma_{\mathrm{ad}(E)}(u_\infty)) \leq 0\,. \tag{2.8.25}$$

The semisum of subsequences $\{\frac{u_i + \sigma_{\mathrm{ad}(E)}u_i}{2}\}$ converges to $u'_\infty = \frac{u'_\infty i + \sigma_{\mathrm{ad}(E)}u'_\infty}{2} \in \mathrm{Met}^p_{2\,B}$ and this element is invariant under $\sigma_{\mathrm{ad}(E)}$. Then $\rho(u'_\infty)$ has real constant eigenvalues. The filtration $V(u'_\infty)$ induces a reduction σ of structure group from G to P such that $E_P = \sigma_E(E_P)$, therefore $\sigma_{\mathrm{Ad}(E)}(\mathrm{Ad}E_{P_s}) = \mathrm{Ad}E_{P_s}$. By hypothesis (A,ϕ,σ_E) is α-stable, then $\deg E(\sigma,s) + B(\alpha,s) > 0$. In view of Proposition 2.8.1, this inequality contradicts (2.8.25).

$\square$

Let $\{s_i\}$ be a minimizing sequence for Ψ^α, in view of Proposition 2.8.3, after passing to a subsequence, we can suppose that $\{s_i\}$ converges weakly to some s that minimize Ψ^α and g is a solution to Theorem 2.8.5.

2.9 Existence of solutions implies polystability

Proposition 2.9.1 (Existence of solutions implies semistability) If there exist a σ_E-compatible Hermite-Einstein-Higgs reduction h of a $(c, \sigma_L, \pm)$-real (quaterionic) Higgs pair (E, φ, σ_E), then it is α-semistable.

Proof: Suppose that (E, φ, σ_E) is not α-semistable. Then there is a reduction σ of structure group of E from G to P_s such that $\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_{P_s} = \mathrm{Ad}E_{P_s}$ and

$$\deg E(s,\sigma) - B(s,\alpha) \;<\; 0\,. \tag{2.9.26}$$

Let A be the connection on E_h that corresponds to the holomorphic structure on E. From Proposition 2.4.1 and Equation (2.9.26), one has

$$\lambda^\alpha(\,(A,\varphi)\,,\sqrt{-1}\epsilon_{h,\sigma,s}\,) \;<\; 0\,. \tag{2.9.27}$$

By hypothesis $\lambda^\alpha_0(\,(A,\varphi)\,,\sqrt{-1}\epsilon_{h,\sigma,s}\,) = 0$ and $\lambda^\alpha_t(\,(A,\varphi)\,,\sqrt{-1}\epsilon_{h,\sigma,s}\,)$ is a non decreasing sequence, therefore $\lambda^\alpha(\,(A,\varphi)\,,\sqrt{-1}\epsilon_{h,\sigma,s}\,) \geq 0$ and this contradicts (2.9.26).

□

Proposition 2.9.2 (Existence of solutions implies polystability) If there exist a σ_E-compatible Hermite-Einstein-Higgs reduction h of a $(c, \sigma_L, \pm)$-real (quaterionic) Higgs pair (E, φ, σ_E), then it is α-polystable.

Proof: Let h be a σ_E-compatible Hermite-Einstein-Higgs reduction. Let $s \in i\mathfrak{h}$ and let σ be any reduction of structure group from G to P_s such that $\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_{P_s} = \mathrm{Ad}E_{P_s}$. Since h is σ_E-compatible, then $\sigma_E(E_h) = E_h$, where E_h is the reduced H-bundle. We define ψ' in the following commutative diagram:

$$\begin{tikzcd} E_h \arrow[r, "\psi"] \arrow[d, "\sigma_E"'] & G/P_s \arrow[d, "\sigma_G"] \\ E_h \arrow[r, "\psi'"] & G/P_{d\sigma_G(s)}\,, \end{tikzcd} \tag{2.9.28}$$

where ψ is defined in (2.4.10). Analogously, we define ξ'

$$\begin{tikzcd} E_h \arrow[r, "\xi"] \arrow[d, "\sigma_E"'] & \sqrt{-1}\mathfrak{h} \arrow[d, "d\sigma_G"] \\ E_h \arrow[r, "\xi'"] & \sqrt{-1}\mathfrak{h}\,, \end{tikzcd} \tag{2.9.29}$$

where ξ is defined in (2.4.10). The maps ξ and ξ' define sections $\epsilon_{h,\sigma,s}$ and $\epsilon'_{h,\sigma,s}$ in $E_h(\sqrt{-1}\mathfrak{h})$ such that the associated reductive filtrations $V(\epsilon)$ and $V(\epsilon')$ are related in the following way: they define reductions E_P and $\sigma_E(E_P)$ and $\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_{P_s} = \mathrm{Ad}E_{P_s}$.

From the previous Proposition, (E, φ, σ_E) is α-semistable. Let σ be a reduction of structure group of E from G to P_s such that $\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}E_{P_s} = \mathrm{Ad}E_{P_s}$ and

$$\deg E(s, \sigma) - B(s, \alpha) = 0\,. \tag{2.9.30}$$

Let A be the connection on E_h that corresponds to the holomorphic structure on E. From Proposition 2.4.1 and Equation (2.9.30), it follows that

$$\lambda^{\alpha}(\,(A, \varphi)\,, \sqrt{-1}\epsilon_{h,\sigma,s}\,) = 0\,. \tag{2.9.31}$$

By hypothesis $\lambda_0^{\alpha}(\,(A, \varphi)\,, \sqrt{-1}\epsilon_{h,\sigma,s}\,) = 0$ and $\lambda_t^{\alpha}(\,(A, \varphi)\,, \sqrt{-1}\epsilon_{h,\sigma,s}\,)$ is a non decreasing sequence, therefore $\lambda_t^{\alpha}(\,(A, \varphi)\,, \sqrt{-1}\epsilon_{h,\sigma,s}\,) = 0$ for any t. This implies that $e^{t\epsilon_{h,\sigma,s}}$ fix A, for any t. So the filtration $V(\epsilon_{h,\sigma,s})$ induces a reduction of the structure group of E_P from P to L. Analogously, we can see that the filtration $e^{t\epsilon'_{h,\sigma,s}}$ fix $\sigma_{\mathcal{A}}A$, for any t, where $\sigma_{\mathcal{A}}$ is the real structure induced in the space of connections by σ_E and σ_X. The filtrations $V(\epsilon_{h,\sigma,s})$ and $V(\epsilon'_{h,\sigma,s})$ induce reductions of the structure group to L and $\sigma_G L$ that are related by

$$\sigma_{\mathrm{Ad}(E)}(\mathrm{Ad}(E_{L_s})) = \mathrm{Ad}(E_{L_s})\,.$$

The element $e^{t\epsilon_{h,\sigma,s}}$ also fix φ, for any t. This implies that $\varphi \in H^0(X, E_{L_s}(\mathbb{V}_s^0) \otimes L)$.
□

2.10 Description of the fixed point locus

Let X be a compact Riemann surface, G be a connected reductive complex Lie group and $\mathbb{V}$ a complex Lie group. Let $\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$ be a representation and L a line bundle over X. Let $\alpha \in \mathfrak{z}(\mathfrak{h})$.

Definition 2.10.1 The **moduli space $\mathcal{M}_{\alpha,d}$ of α-polystable Higgs pairs of degree** d is the set of isomorphism classes of α-polystable Higgs pairs (E, φ) such that the degree of E is d.

Let α be an element of $\mathfrak{z}(\mathfrak{h})$ invariant under $d\sigma_G$. Thanks to Theorem 2.5.1, we define the following involutions on $\mathcal{M}_{\alpha,d}$ depending on: a real stucture σ_X on X, a real stucture σ_G on G, a real (quaternionic) structure $\sigma_{\mathbb{V}}$ on $\mathbb{V}$ such that ρ is $(\sigma_G, \sigma_{\mathbb{V}})$-real (quaternionic), a real (quaternionic) stucture σ_L on L and a parameter $\pm$.

$$\begin{array}{rccc} i_{\mathcal{M}_{\alpha,d}}(\sigma_X, \sigma_G, \sigma_{\mathbb{V}}, \sigma_L)^{\pm} : & \mathcal{M}_{\alpha,d} & \longrightarrow & \mathcal{M}_{\alpha,d} \\ & (E, \varphi) & \longmapsto & (\,(\sigma_X)^*\sigma_G E\,, \pm\phi_{E,\sigma_{\mathbb{V}},\sigma_L}(\varphi)\,), \end{array}$$

where

$$\phi_{E,\sigma_{\mathbb{V}},\sigma_L} : E(\mathbb{V}) \otimes L \to (\sigma_X)^*\sigma_G E(\mathbb{V}) \otimes L \qquad (2.10.32)$$

is the map induced by $\phi_E, \sigma_{\mathbb{V}}$ and σ_L, with $\phi_E : E \to \sigma_X^*\sigma_G E$ being the holomorphic isomorphism of bundles induced by σ_E. (Proposition 1.6.1).

Definition 2.10.2 Let $c \in Z_2^{\sigma_G} \cap \ker\rho$. The **moduli space** $\mathcal{M}_{\alpha,d}(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)$ **of** α**-polystable** $(c, \sigma_L, \pm)$**-real (quaternionic) Higgs pairs of degree** d is the set of isomorphism classes of α-polystable $(c, \sigma_L, \pm)$-real Higgs pairs (E, φ, σ_E) such that the degree of E is d.

From Corollary 2.5.2, it follows that there is an embedding of $\mathcal{M}_{\alpha,d}(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)$ into $\mathcal{M}_{\alpha,d}$. Let us denote by $\widetilde{\mathcal{M}_{\alpha,d}}(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)$ the image of this embedding.

Definition 2.10.3 The **deformation complex of Higgs pair** (E, φ) is the following complex of sheaves:

$$C^\bullet(E, \varphi) : E(\mathfrak{g}) \xrightarrow{d\rho(\varphi)} E(\mathbb{V}) \otimes K \ . \qquad (2.10.33)$$

Definition 2.10.4 The **smooth locus** $(\mathcal{M}_{\alpha,d})_{\mathrm{sm}}$ **of the moduli space** $\mathcal{M}_{\alpha,d}$ is the set of Higgs pairs (E, φ) of degree d that are α-stable, simple and $\mathbb{H}^2(C^\bullet(E, \varphi)) = 0$.

Definition 2.10.5 The **smooth locus** $\mathcal{M}_{\alpha,d}(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)_{\mathrm{sm}}$ **of the moduli space** $\mathcal{M}_{\alpha,d}(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)$ is the set of $(c, \sigma_L, \pm)$-real (quaternionic) Higgs pairs (E, φ, σ_E) of degree d that are α-stable, simple and $\mathbb{H}^2(C^\bullet(E, \varphi)) = 0$.

We denote by $\widetilde{\mathcal{M}_{\alpha,d}}(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)_{sm}$ the image of $\mathcal{M}_{\alpha,d}(\sigma_X, \sigma_G, c, \sigma_L, \sigma_{\mathbb{V}}, \pm)_{\mathrm{sm}}$.

Proposition 2.10.1 The fixed points of the involution $i_{\mathcal{M}}(\sigma_X, \sigma_G, \sigma_{\mathbb{V}}, \sigma_L)^{\pm}$ and the moduli space of α-polystable $(c, \sigma_L, \pm)$-real (quaternionic) Higgs pair are related by:

(1)

$$(\mathcal{M}_{\alpha,d})^{i_{\mathcal{M}_{\alpha,d}}(\sigma_X,\sigma_G,\sigma_{\mathbb{V}},\sigma_L)^{\pm}} \supseteq \bigcup_{c\in Z_2^{\sigma_G}\cap\ker(\rho)} \widetilde{\mathcal{M}_{\alpha,d}}(\sigma_X,\sigma_G,c,\sigma_L,\sigma_{\mathbb{V}},\pm)\,,$$

(2) For $g(X)\geq 2$ and if we restrict to the smooth locus sm, then

$$(\mathcal{M}_{\alpha,d})_{\mathrm{sm}}^{i_{\mathcal{M}_{\alpha,d}}(\sigma_X,\sigma_G,\sigma_{\mathbb{V}},\sigma_L)^{\pm}} \cong \bigcup_{c\in Z_2^{\sigma_G}\cap\ker(\rho)} \widetilde{\mathcal{M}_{\alpha,d}}(\sigma_X,\sigma_G,c,\sigma_L,\sigma_{\mathbb{V}},\pm)_{\mathrm{sm}}\,.$$

Proof:

(1) If $(E,\varphi)\in\widetilde{\mathcal{M}_{\alpha,d}}(\sigma_X,\sigma_G,c,\sigma_L,\sigma_{\mathbb{V}},\pm)$ is the image of $(E,\varphi,\sigma_E)\in\mathcal{M}_{\alpha,d}(\sigma_X,\sigma_G,c,\sigma_L,\sigma_{\mathbb{V}},\pm)$, then there is a holomorphic isomorphism of bundles $\phi_E: E\to\sigma_X^*\sigma_G E$ that is induced by σ_E. (Proposition 1.6.1) Using ϕ_E, we can define $\phi_{E,\sigma_{\mathbb{V}},\sigma_L}$ as in Equation (2.10.32), then

$$i_{\mathcal{M}_{\alpha,d}}(\sigma_X,\sigma_G,\sigma_{\mathbb{V}},\sigma_L)^{\pm}(E,\varphi)\cong(\phi_E(E),\pm\phi_{E,\sigma_{\mathbb{V}},\sigma_L}(\varphi)\cong(E,\varphi)\,,$$

where the last isomorphism is true because $\sigma_{E^L(\mathbb{V})}\varphi=\pm\varphi$.

(2) Let $(E,\varphi)\in\mathcal{M}_{\mathrm{sm}}^{i_{\mathcal{M}}(\sigma_X,\sigma_G)^{\pm}}$. There is an isomorphism

$$\phi_E: E\to\sigma_X^*\sigma_G E\,,$$

such that $(\phi_E(E),\pm\phi_{E,\sigma_{\mathbb{V}},\sigma_L}\varphi)\cong(E,\varphi)$. The composition $\sigma_X^*\sigma_G\phi_E\circ\phi_E$ belongs to $\mathrm{Aut}(E,\varphi)$ that is equal to $Z\cap\mathrm{Ker}(\rho)$ because (E,φ) is simple. Let $c:=\sigma_X^*\sigma_G(\phi_E)\circ\phi_E\in Z\cap\mathrm{Ker}(\rho)$. Since ϕ_E commutes with $\sigma_X^*\sigma_G\phi_E\circ\phi_E$ then $\sigma_G(c)=c$. Then $c\in Z_2^{\sigma_G}\cap\ker(\rho)$. From Proposition 1.6.1, there is a (σ_X,σ_G,c)-real structure σ_E on E. The real structure $\sigma_{E^L(\mathbb{V})}$ induced by σ_E, σ_L and $\sigma_{\mathbb{V}}$ satisfies that $\sigma_{E^L(\mathbb{V})}(\varphi)=\pm\varphi$, because by hypothesis $\varphi\cong\pm\phi_{E,\sigma_{\mathbb{V}},\sigma_L}(\varphi)$.

$\square$

Chapter 3

Real G-Higgs bundles and non-abelian Hodge correspondence

Along this chapter, we use the following notation:

- $G-$real form of a semisimple Lie group $G^{\mathbb{C}}$
- $\mu-$conjugation of $G^{\mathbb{C}}$ such that $(G^{\mathbb{C}})^{\mu} = G$
- $\tau-$conjugation of $G^{\mathbb{C}}$ such that $(G^{\mathbb{C}})^{\tau} = H^{\mathbb{C}}$ is a maximal compact subgroup of $G^{\mathbb{C}}$
- $\theta := \sigma \circ \tau$
- σ_G-conjugation of $G^{\mathbb{C}}$ that is (μ, τ)-compatible
- $Z(G^{\mathbb{C}}), Z(H^{\mathbb{C}})-$centers of $G^{\mathbb{C}}, H^{\mathbb{C}}$, respectively
- $Z(H^{\mathbb{C}})_2^{\sigma_G}-$subgroup of elements of $Z(H^{\mathbb{C}})$ of order two that are invariants under σ_G
- $\mathfrak{g}, \mathfrak{h}, \mathfrak{z}-$Lie algebras of G, H, Z, respectively
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}-$Cartan decomposition
- $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})-$isotropy representation
- $c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap \mathrm{Ker}(\iota)$

- $\mathcal{M}_d$−moduli space of polystable G-Higgs bundles of characteristic class d
- $\mathcal{S}$−moduli space of reductive local systems
- $\mathcal{R}$−moduli space of reductive representations of $\pi_1(X)$ into G.

3.1 Real G-Higgs bundles

Let G be a semisimple real form of a connected reductive Lie group $G^{\mathbb{C}}$ and $\mu \in \mathrm{Conj}(G^{\mathbb{C}})$ be a conjugation such that the fixed point set $(G^{\mathbb{C}})^{\mu}$ of $G^{\mathbb{C}}$ under μ is equal to G. Let $\tau \in \mathrm{Conj}(G^{\mathbb{C}})$ such that $(G^{\mathbb{C}})^{\tau} = H^{\mathbb{C}}$ is a maximal compact subgroup of $G^{\mathbb{C}}$. Let σ_G be a (μ,τ)-compatible conjugation of $G^{\mathbb{C}}$. (Definition 1.3.10). From Proposition 1.3.5, $d\theta := d(\sigma \circ \tau)$ is a Cartan involution for Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the Cartan decomposition. Since $d\sigma_G$ commutes with $d\theta$, then $d\sigma_G$ preserves $\mathfrak{h}$ and $\mathfrak{m}$. We denote, by abuse of notation, σ_G to the restriction of $d\sigma_G$ to this spaces. The adjoint action of H on $\mathfrak{m}$ extends to the complexified spaces, giving the **isotropy representation**

$$\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}}), \tag{3.1.1}$$

that is $(\sigma_G, d\sigma_G)$-real (Definition 1.4.3). Let (X, σ_X) be a compact Klein surface and K be the canonical bundle. The antiholomorphic involution σ_X induces a real structure σ_K on K. Let $Z(G^{\mathbb{C}})$ and $Z(H^{\mathbb{C}})$ be the centers of $G^{\mathbb{C}}, H^{\mathbb{C}}$, respectively. Let $Z(H^{\mathbb{C}})_2^{\sigma_G}$ be the subgroup of elements of $Z(H^{\mathbb{C}})$ of order two that are invariants under σ_G. Since the kernel of the adjoint representation is $Z(G^{\mathbb{C}})$, then

$$Z(H^{\mathbb{C}})_2^{\sigma_G} \cap \ker \iota = Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}}) \cap H^{\mathbb{C}} = Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}}). \tag{3.1.2}$$

Definition 3.1.1 Let $c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})$. A $(\sigma_X, \sigma_G, c, \pm)$-**real G-Higgs bundle** is a $(\sigma_X, \sigma_G, c, d\sigma_G, \sigma_K, \pm)$-real $H^{\mathbb{C}}$-Higgs-pair. (Definition 2.1.2).

Let $\mathfrak{z}(\mathfrak{h}^{\mathbb{C}})$ be the center of $\mathfrak{h}^{\mathbb{C}}$ and $\alpha \in i\mathfrak{h} \cap \mathfrak{z}(\mathfrak{h}^{\mathbb{C}})$. The notions of α-stability, α-semistability and α-polystability for $(\sigma_X, \sigma_G, c, \pm)$-real G-Higgs bundles are the same that for real $H^{\mathbb{C}}$-Higgs-pairs (see Section 2.3). For the purpose of proving the non-abelian Hogde correspondence, we consider G to be semisimple. In this case

$\alpha = 0$, that is why we use the following terminology: a $(\sigma_X, \sigma_G, c, \pm)$-real G-Higgs bundle (E, φ, σ_E) is **stable** if it is 0-stable. We define semistability and polystability similarly.

We denote by $\mathcal{M}_d(\sigma_X, \sigma_G, c, \pm)$ the moduli space of equivalence classes of polystable $(\sigma_X, \sigma_G, c, \pm)$-real G-Higgs bundles (E, φ, σ_E) with characteristic class $d = c(E) \in \pi_1(G)$.

3.2 Hitchin equations

Let (E, φ, σ_E) be a $(\sigma_X, \sigma_G, c, \pm)$-real G-Higgs bundle over a compact Klein surface (X, σ_X). Let h be a σ_E-compatible reduction of the structure group from $H^{\mathbb{C}}$ to H (See Definition 1.6.4) and let

$$\tau_h : \Omega^{1,0}(E(\mathfrak{g}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{g}^{\mathbb{C}})) \tag{3.2.3}$$

be the map given by the compact conjugation τ in the fibres induced by h combined with complex conjugation on complex 1-forms.

Theorem 3.2.1 (Hitchin-Kobayashi correspondence) A $(\sigma_X, \sigma_G, c, \pm)$-real G-Higgs bundle (E, φ, σ_E) is polystable if and only if there exist a σ_E-compatible reduction h of the structure group of E from $H^{\mathbb{C}}$ to H satisfying the Hermite-Einstein-Higgs equation

$$F_h - [\varphi, \tau_h(\varphi)] = 0\,. \tag{3.2.4}$$

Proof: Is a particular case of the Hitchin-Kobayashi correspondence for α-polystable Higgs-pairs proved in Theorem 2.5.1. $\square$

Corollary 3.2.2 A $(\sigma_X, \sigma_G, c, \pm)$-real G Higgs bundle (E, φ, σ_E) is polystable if and only if the underlying G-Higgs bundles (E, φ) is polystable.

Proof: It is a particular case of Corollary 2.5.2. $\square$

From Theorem 1.6.2, there is an equivalence between real principal bundles and real holomorphic structures, therefore the moduli space $\mathcal{M}_d(\sigma_X, \sigma_G, c, \pm)$ is equivalent to the set of triples $(\bar{\partial}_E, \varphi, \sigma_{\mathbb{E}})$ modulo $\mathcal{G}^{\sigma_{\mathcal{G}}}$, where

(1) $\bar{\partial}_E$ is a (σ_G, σ_E)-real holomorphic structure on the C^∞ bundle $\mathbb{E}$,

(2) $\sigma_{\mathbb{E}}$ is a (σ_X, σ_G, c)-real structure on $\mathbb{E}$ such that:

- $\bar{\partial}_E(\varphi) = 0$,
- if E the holomorphic bundle corresponding to $\bar{\partial}_E$ and σ_E the real structure on E induced by $\sigma_{\mathbb{E}}$ then (E, φ, σ_E) is a polystable $(\sigma_X, \sigma_G, c, \pm)$-real G-Higgs bundle.

(3) the Higgs field $\varphi \in \Omega^{1,0}(\mathbb{E}_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$ is fixed under the involution on $\Omega^{1,0}(\mathbb{E}_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$ induced by $\sigma_{\mathbb{E}}, \sigma_G$ and σ_X.

(4) $\mathcal{G}$ is the gauge group of $\mathbb{E}$, $\sigma_{\mathcal{G}}$ is the involution induced by $\sigma_{\mathbb{E}}$ and σ_G and $\mathcal{G}^{\sigma_{\mathcal{G}}}$ is the fixed point set of $\mathcal{G}$ under $\sigma_{\mathcal{G}}$.

Let $\mathcal{A}_d(\sigma_X, \sigma_G, c, \pm)$ be the moduli space given by triples $(A, \varphi, \sigma_{\mathbb{E}})$ modulo $\mathcal{H}^{\sigma_{\mathcal{H}}}$ where:

(1) A is a (σ_X, σ_G)-compatible H-connection on $\mathbb{E}_H$ (Proposition 1.6.3)

(2) the Higgs field $\varphi \in \Omega^{1,0}(\mathbb{E}_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$ is fixed under the involution on $\Omega^{1,0}(\mathbb{E}_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$ induced by $\sigma_{\mathbb{E}}, \sigma_G$ and σ_X.

(3) $\sigma_{\mathbb{E}}$ is a (σ_X, σ_G, c)-real structure on $\mathbb{E}$ such that

$$\begin{aligned} \bar{\partial}_A(\varphi) &= 0 \\ F_A - [\varphi, \tau(\varphi)] &= 0\,. \end{aligned} \tag{3.2.5}$$

(4) $\mathcal{H}$ is the gauge group of $\mathbb{E}_H$, $\sigma_{\mathcal{H}}$ is the involution induced by $\sigma_{\mathbb{E}}$ and σ_G and $\mathcal{H}^{\sigma_{\mathcal{H}}}$ is the fixed point set of $\mathcal{H}$ under $\sigma_{\mathcal{H}}$.

In view of Theorem 1.6.4, there is a bijective correspondence between compatible h-connections and real Dolbeaut structures, therefore the moduli space $\mathcal{M}_d(\sigma_X, \sigma_G, c, \pm)$ is in one-to-one correspondence with $\mathcal{A}_d(\sigma_X, \sigma_G, c, \pm)$.

Taking $\phi = \varphi - \tau(\varphi)$, the moduli space $\mathcal{A}_d(\sigma_X, \sigma_G, c, \pm)$ is equivalent to the space of triples $(A, \phi, \sigma_{\mathbb{E}})$ modulo $\mathcal{H}^{\sigma_{\mathcal{H}}}$ satisfying the **Hitchin equations**:

$$\begin{aligned} F_A - [\phi, \tau(\phi)] &= 0 \\ d_A(\phi) &= 0 \\ d_A^*(\phi) &= 0\,. \end{aligned} \tag{3.2.6}$$

3.3 Compatible local systems and representations

We extend the definition of $\sigma_{\mathbb{E}}$-compatible connection, given in (4) of Proposition 1.6.3) in the following way.

Definition 3.3.1 A connection D on a C^∞ principal G-bundle $\mathbb{E}$ over X is $(\sigma_{\mathbb{E}}, \pm)$-**compatible** if it satisfies

$$d(\sigma_G) \circ D \circ \sigma_{\mathbb{E}} = \pm D\,.$$

Remark 3.3.1 A $(\sigma_{\mathbb{E}}, +)$-compatible connection is just a $\sigma_{\mathbb{E}}$-compatible one.

Definition 3.3.2 Let $c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})$. A $(\sigma_X, \sigma_G, c, \pm)$-**compatible local system** is a triple $(\mathbb{E}, \sigma_{\mathbb{E}}, D)$, where $\mathbb{E}$ is a C^∞ a principal bundle, $\sigma_{\mathbb{E}}$ is a (σ_X, σ_G, c)-real structure on $\mathbb{E}$ and D is a $(\sigma_{\mathbb{E}}, \pm)$-compatible flat connection.

Two $(\pm, c)$-compatible local systems $(\mathbb{E}, \sigma_{\mathbb{E}}, D)$ and $(\mathbb{E}', \sigma_{\mathbb{E}'}, D')$ are equivalent if there is an $(\sigma_{\mathbb{E}}, \sigma_{\mathbb{E}'})$-compatible automorphism $\phi : \mathbb{E} \to \mathbb{E}'$ such that $\phi^* D' = D$.

Definition 3.3.3 Let $S(\sigma_X, \sigma_G, c, \pm)$ be the set of equivalence classes of $(\pm, c)$-compatible local systems.

Definition 3.3.4 ([17], Section 5.1.) Fix a point $x \in X$ such that $\sigma_X(x) \neq x$. The **orbifold fundamental group** $\Gamma(X, x)$ of (X, σ_X) is, as a set, the disjoint union of $\pi_1(X, x)$ and

$$\mathrm{Path}(X, x) := \{\text{homotopy classes of paths } \gamma : [0,1] \to X \mid \gamma(0) = x,\, \gamma(1) = \sigma_X(x)\},$$

with the composition rule $\gamma_2\gamma_1 = \sigma_X^q(\gamma_2) \circ \gamma_1$, where

$$q = \begin{cases} 0 & \text{if } \gamma_1 \in \pi_1(X, x) \\ 1 & \text{if } \gamma_1 \in \mathrm{Path}(X, x)\,. \end{cases}$$

The function q induces the short exact sequence

$$0 \longrightarrow \pi_1(X,x) \xrightarrow{i} \Gamma(X,x) \xrightarrow{q} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \ ,$$

where i denotes the inclusion of groups.

Let $\widehat{G}^{\pm}$ be two possible groups depending on a parameter $\pm$. As a set $\widehat{G}^{\pm} = G \times (\mathbb{Z}/2\mathbb{Z})$ and the group operation depending on the parameter $\pm$ is given by

$$(g_1, e_1)(g_2, e_2) = (g_1(\sigma_G \tau^{-\frac{1}{2}\pm\frac{1}{2}})^{e_1}(g_2)c^{e_1 e_2}, e_1 + e_2)\,.$$

Definition 3.3.5 A $(\sigma_X, \sigma_G, c, \pm)$-**compatible representation** of $\Gamma(X,x)$ is a map

$$\rho \,:\, \Gamma(X,x) \longrightarrow \widehat{G}^{\pm}$$

that fits in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(X,x) \arrow[r, "i"] \arrow[d] & \Gamma(X,x) \arrow[r, "q"] \arrow[d, "\rho"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & G \arrow[r, "i"] & \widehat{G}^{\pm} \arrow[r, "q"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0\,, \end{tikzcd} \tag{3.3.7}$$

and satisfies:

(1) the restriction of ρ to $\pi_1(X,x)$ is a homomorphism of groups,

(2) $\rho(g'g) = c\rho(g)\rho(g')$, if $q(g) \cdot q(g') = 1\,,$

(3) $\rho(g'g) = \rho(g)\rho(g')$, if $q(g) \cdot q(g') = 0\,.$

We denote by $\mathrm{Hom}_c(\Gamma, \widehat{G}^{\pm})$ the set of $(\sigma_X, \sigma_G, c, \pm)$-compatible representation of $\Gamma(X,x)$. The quotient set $\mathrm{Hom}_c(\Gamma, \widehat{G}^{\pm})/G$ by conjugations will be denoted by $R(\sigma_X, \sigma_G, c, \pm)$.

Theorem 3.3.6 There is a bijective correspondence between $S(\sigma_X, \sigma_G, c, \pm)$ and $R(\sigma_X, \sigma_G, c, \pm)$.

Proof: The correspondence without real structures is proved in [82, Section 13.9]. Let $(\mathbb{E}, \sigma_{\mathbb{E}}, D)$ be a $(\sigma_X, \sigma_G, c, \pm)$-compatible local system, the holonomy $\mathrm{hol}(D)$ of D is a representation $\mathrm{hol}(D) : \pi_1(X,x) \to G$. Now we want to extend $\mathrm{hol}(D)$ to a $(\sigma_X, \sigma_G, c, \pm)$-real representation ρ of $\Gamma(X, x_0)$. In view of Remark [1.6.1], one has that a (σ_X, σ_G, c)-real structure $\sigma_{\mathbb{E}}$ on $\mathbb{E}$ is equivalent to a map $\phi_{\mathbb{E}} : \mathbb{E} \to \sigma_X^* \sigma_G(\mathbb{E})$ such that $\sigma_X^* \sigma_G(\phi_{\mathbb{E}}) \circ \phi_{\mathbb{E}} = c\mathrm{Id}_{\mathbb{E}}$. Let z_0 be an element in the fiber $\mathbb{E}_{x_0}$. Let $\gamma' \in \pi_1(X, x_0)$ be any closed path, let $z_{\gamma'}$ be the point obtained by parallel transport of z_0 along γ' for the connection D. Let $g_{\gamma'} \in G$ be the element of the group such that

$$z_0 g_{\gamma'}^{-1} = z_\gamma \,.$$

Let $\gamma'' \in \Gamma(X, x_0) \setminus \pi_1(X, x_0)$. Let $y \in \mathbb{E}$ be the point obtained by parallel transportation of z_0 along γ'' for D. Let $g_{\gamma''} \in G$ be the element of the group such that

$$z_0 = \phi_{\mathbb{E}}(y) \sigma_G(g_{\gamma'}) \,. \tag{3.3.8}$$

We define $\rho(\gamma) := g_\gamma$, for every $\gamma \in \Gamma(X, x_0)$. Following the same arguments that in the proof of [15, Proposition 4.4], we have that ρ is a $(\sigma_X, \sigma_G, c, \pm)$-real representation. Equivalence classes of $(\pm, c)$-compatible local system correspond to equivalence classes of real representations because $\mathrm{hol}(D)$ satisfies this property:

$$\mathrm{hol}(D)(eg^{-1}) = g^{-1}\mathrm{hol}(D)(e)g \,,$$

for all $g \in G$ and for all $e \in \mathbb{E}_{x_0}$.

Conversely, a $(\sigma_X, \sigma_G, c, \pm)$-real representation restricted to $\pi_1(X, x_0)$ corresponds to a local system $(\mathbb{E}, D)$. A map $\phi_{\mathbb{E}} : \mathbb{E} \to \sigma_X^* \sigma_G(\mathbb{E})$ can be constructed using (3.3.8) and we can extend to the other fibres as in the same way that it is done in [15, p. 18]. From [15, Proposition 4.3], the morphism $\phi_{\mathbb{E}}$ defines a (σ_X, σ_G, c)-real structure on $\mathbb{E}$. The connection D is $(\sigma_{\mathbb{E}}, +)$-compatible by construction for $\sigma_{\mathbb{E}}$. □

3.4 Corlette-Donaldson correspondence

Definition 3.4.1 A **local system** is a pair consisting of a C^∞ a principal bundle $\mathbb{E}$ over X and a flat connection. Two local systems $(\mathbb{E}, D)$ and $(\mathbb{E}', D')$ are related if and only if there is an automorphism $\phi : \mathbb{E} \to \mathbb{E}'$ such that $\phi^* D' = D$.

$$\begin{array}{rccc} i_{\mathcal{S}}(\sigma_X, \sigma_G)^{\pm} : & \mathcal{S} & \longrightarrow & \mathcal{S} \\ & D & \longmapsto & \sigma_X^*(\sigma_G \circ \tau^{\frac{1}{2} \mp \frac{1}{2}}(D)) \,. \end{array} \tag{3.4.9}$$

In Theorem 3.3.6 of the previous section we proved that there exist a bijective correspondence between the moduli space $S(\sigma_X, \sigma_G, c, \pm)$ of $(\pm, c)$-real flat $H^{\mathbb{C}}$-connections and the moduli space $R(\sigma_X, \sigma_G, c, \pm)$ of representations of orbifold fundamental group in $H^{\mathbb{C}}$. Now, we restrict these moduli spaces for reductive objets.

Definition 3.4.2 A morphism $\rho : \pi_1(X) \to G$ is **reductive** if its composition with the adjoint representation (Definition 1.3.8) decomposes in sum of irreducible representations.

Definition 3.4.3 A $(\sigma_X, \sigma_G, c, \pm)$-compatible representation of the orbifold fundamental group $\rho : \Gamma(X, x) \longrightarrow \widehat{G}^{\pm}$ is **reductive** if its restriction to $\pi_1(X)$ is reductive.

Definition 3.4.4 A $(\sigma_X, \sigma_G, c, \pm)$-compatible local system $(\mathbb{E}, \sigma_{\mathbb{E}}, D)$ is **reductive** if it comes from a reductive representation, via the homeomorphism of Theorem 3.3.6.

We denote by $\mathcal{S}(\sigma_X, \sigma_G, c, \pm)$ the moduli space of reductive $(\sigma_X, \sigma_G, c, \pm)$-compatible local systems and we denote by $\mathcal{R}(\sigma_X, \sigma_G, c, \pm)$ to the moduli space of reductive $(\sigma_X, \sigma_G, c, \pm)$-compatible representations of the orbifold fundamental group.

Remark 3.4.1 From Theorem 3.3.6 and the definitions above, there is a bijective correspondence between $\mathcal{S}(\sigma_X, \sigma_G, c, \pm)$ and $\mathcal{R}(\sigma_X, \sigma_G, c, \pm)$.

Every G-connection D decomposes uniquely as

$$D = d_A + \phi\,,$$

where d_A is an H-connection on $\mathbb{E}_H$ and $\phi \in \Omega^1(X, \mathbb{E}_H(\mathfrak{m}))$.

Theorem 3.4.5 (Corlette-Donaldson correspondence) There is a bijective correspondence between $\mathcal{S}(\sigma_X, \sigma_G, c, \pm)$ and $\mathcal{A}_0(\sigma_X, \sigma_G, c, \pm)$. More, specifically, given a reductive flat $(\sigma_{\mathbb{E}}, \pm)$-compatible G-connection D_0, there exist an element $g \in \mathcal{H}^{\sigma_{\mathcal{H}}}$ such that if $D = g(D_0)$ decomposes as $d_A + \phi$, then the triple $(d_A, \phi, \sigma_{\mathbb{E}})$ satisfies the Hitchin equations (3.2.6).

Proof: The first two equations of (3.2.6) are equivalent to the flatness of D. From the conventional Theorem of Corlette-Donaldson, there is $g \in \mathcal{H}$ such that the reduction $g(h_0)$ induced a map that is $\tilde{X} \to G/H$ is harmonic. But $\sigma_{\mathcal{H}}(g)(h_0)$ induced also a map $\sigma_X(\tilde{X}) \to \sigma_G(G/H)$ which is also harmonic. Since G is semisimple, by unicity $g = \sigma_{\mathcal{H}}(g)$, hence $g \in \mathcal{H}^{\sigma_{\mathcal{H}}}$ satisfies the conditions of the statement. □

Theorem 3.4.6 (Non-abelian Hodge correspondence for real G-Higgs bundles) There is bijective correspondence between $\mathcal{M}(\sigma_X, \sigma_G, c, \pm)$ and $\mathcal{R}(\sigma_X, \sigma_G, c, \pm)$.

Proof: In view of Theorem 3.2.1, it follows that $\mathcal{M}(\sigma_X, \sigma_G, c, \pm)$ is in one-to-one correspondence with $\mathcal{A}_0(\sigma_X, \sigma_G, c, \pm)$, which is in bijection with $\mathcal{S}(\sigma_X, \sigma_G, c, \pm)$ by Theorem 3.4.5. Finally, there is a bijection between $\mathcal{S}(\sigma_X, \sigma_G, c, \pm)$ and $\mathcal{R}(\sigma_X, \sigma_G, c, \pm)$ by Remark 3.4.1. □

3.5 Involutions of the moduli spaces

We consider the holomorhic $H^{\mathbb{C}}$-bundle

$$E' := \sigma_X^* E \times_{\sigma_G} H^{\mathbb{C}}\,.$$

It fits in the following diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \sigma_X^* E \arrow[r, "\pi"] \arrow[d] & E' \arrow[d] \\ X \arrow[r, "\sigma_X"] & X \arrow[r, "\mathrm{Id}"] & X, \end{tikzcd}$$

where $\pi(e) := (e, \mathrm{Id}_{H^{\mathbb{C}}}) \in E'$, for all $e \in E$. We denote by f the composition of the pullback of σ_X and π. The previous diagram can be expressed equivalently as

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\ X \arrow[r, "\sigma_X"] & X\,. \end{tikzcd} \tag{3.5.10}$$

The conjugation σ_G and f induce a lift over σ_X

$$\mathrm{Ad}(f) : E(H^{\mathbb{C}}) \to E'(H^{\mathbb{C}})\,. \tag{3.5.11}$$

The map induced by the tensor product of $\mathrm{Ad}(f)$ and the antiholomorphic involution σ_K in the canonical bundle K, transform a Higgs field $\varphi \in \Gamma(E'(H^{\mathbb{C}}) \otimes K)$ in other Higgs field $\varphi' \in \Gamma(E'(H^{\mathbb{C}}) \otimes K)$. The morphism f preserves the degree and one can see that f sends a polystable bundle to a polystable bundle, so f define involutions in the moduli space $\mathcal{M}$ of classes of polystable G-Higgs bundles of degree 0

$$\begin{array}{rccc} i_{\mathcal{M}}(\sigma_X, \sigma_G)^{\pm}: & \mathcal{M} & \longrightarrow & \mathcal{M} \\ & (E, \varphi) & \longmapsto & (E', \pm\varphi') \,. \end{array} \tag{3.5.12}$$

Definition 3.5.1 A local system is **reductive** if the representation of the fundamental group from which it comes, is reductive (Definition 3.4.2).

We denote by $\mathcal{S}$ the moduli space of reductive local systems. We denote by $\bar{\partial}_E$ the Dolbeaut operator associated to a $H^{\mathbb{C}}-$bundle E. We consider the homeomorphism of moduli spaces

$$\begin{array}{rccc} D: & \mathcal{M} & \xrightarrow{\cong} & \mathcal{S} \\ & (E, \varphi) & \longmapsto & D_{(E,\varphi)} := \bar{\partial}_E + \tau(\bar{\partial}_E) + \varphi - \tau(\varphi) \,. \end{array} \tag{3.5.13}$$

Proposition 3.5.1 The following diagram commute

$$\begin{tikzcd} \mathcal{M} \arrow[r, "D"] \arrow[d, "i_{\mathcal{M}}(\sigma_X,\sigma_G)^{\pm}"'] & \mathcal{S} \arrow[d, "i_{\mathcal{S}}(\sigma_X,\sigma_G)^{\pm}"] \\ \mathcal{M} \arrow[r, "D"] & \mathcal{S}. \end{tikzcd}$$

Proof: Indeed, for all $(E, \varphi) \in \mathcal{M}$,

$$\begin{aligned} D \circ i_{\mathcal{M}}(\sigma_X, \sigma_G)^{+}(E, \varphi) &= D_{(\sigma_X^*\sigma_G E\,,\, \sigma_X^*\sigma_G\varphi)} = \\ &= \sigma_X^* D_{(\sigma_G E, \sigma_G \varphi)} = \sigma_X^*(\sigma_G\bar{\partial}_E + \sigma_G\tau\bar{\partial}_E + \sigma_G\varphi - \sigma_G\tau\varphi) = \\ &= i_{\mathcal{S}}^{+}(\sigma_X, \sigma_G)(D_{(E,\varphi)}) = i_{\mathcal{S}}^{+}(\sigma_X, \sigma_G) \circ D(E, \varphi) \,. \end{aligned} \tag{3.5.14}$$

Recall that $\theta = \sigma \circ \tau$. We consider

$$\begin{array}{rccc} i_{\mathcal{M}}(\theta)^{\pm}: & \mathcal{M} & \longrightarrow & \mathcal{M} \\ & (E, \varphi) & \longmapsto & (\theta E\,,\, \pm\theta\varphi) \,. \end{array}$$

The relation between $i_{\mathcal{M}}(\sigma_X, \sigma_G)^{-}$ and $i_{\mathcal{M}}(\theta)^{\pm}$ is given by

$$i_{\mathcal{M}}(\sigma_X, \sigma_G)^{-} = i_{\mathcal{M}}(\theta)^{-} \circ i_{\mathcal{M}}(\sigma_X, \tau)^{+} \,. \tag{3.5.15}$$

On the other hand, we define $i_{\mathcal{S}}(\theta)^-(D_{(E,\varphi)}) := D_{i_{\mathcal{M}}(\theta)^-(E,\varphi)}$, then

$$\begin{aligned} i_{\mathcal{S}}(\theta)^- \circ D_{(E,\varphi)} &= D_{(\theta E\,,\theta\varphi)} = \theta\bar{\partial}_E + \theta\tau(\bar{\partial}_E) - \theta\varphi + \theta\tau(\varphi) = \\ &= \sigma_G\tau\bar{\partial}_E + \sigma_G(\bar{\partial}_E) - \sigma_G\tau\varphi + \sigma_G\tau(\varphi) = \sigma_G(D_{(E,\varphi)})\,. \end{aligned} \tag{3.5.16}$$

By Equation (3.5.16) above, we obtain the equality

$$i_{\mathcal{S}}(\sigma_X,\sigma_G)^- = i_{\mathcal{S}}(\theta)^- \circ i_{\mathcal{S}}(\sigma_X,\tau)^+\,. \tag{3.5.17}$$

Finally, In view of (3.5.17), (3.5.14) and the definition of $i_{\mathcal{S}}(\theta)^-$, one has

$$\begin{aligned} i_{\mathcal{S}}(\sigma_X,\sigma_G)^- \circ D(E,\varphi) &= i_{\mathcal{S}}(\theta)^- \circ i_{\mathcal{S}}(\sigma_X,\tau)^+ \circ D(E,\varphi) = \\ &= i_{\mathcal{S}}(\theta)^- \circ D \circ i_{\mathcal{M}}(\sigma_X,\tau)^+(E,\varphi) = i_{\mathcal{S}}(\theta)^-(D_{(\sigma_X^*\tau E,\sigma_X^*\tau\varphi)}) = \\ &= D_{i_{\mathcal{M}}(\theta)^-(\sigma_X^*\tau E,\sigma_X^*\tau\varphi)} = D_{(\sigma_X^*\sigma_G E\,,\sigma_X^*\sigma_G\varphi)} = D\circ i_{\mathcal{M}}(\sigma_X,\sigma_G)^-(E,\varphi)\,. \end{aligned} \tag{3.5.18}$$

□

Let $\mathrm{Hom}^+(\pi_1(X),G)$ be the subspace of $\mathrm{Hom}(\pi_1(X),G)$ consisting of reductive representations (Definition 3.4.2). We denote by $\mathcal{R}$ the moduli space $\mathrm{Hom}^+(\pi_1(X),G)/G$. Let $(\sigma_X)_* : \pi_1(X) \to \pi_1(X)$ be the involution on $\pi_1(X)$ given by

$$(\sigma_X)_*\gamma(t) := \sigma_X(\gamma(t)),\ \text{ for every } \gamma\in\pi_1(X), t\in[0,1]\,.$$

Consider the involutions

$$\begin{aligned} i_{\mathcal{R}}(\sigma_X,\sigma_G)^\pm :\ \mathcal{R} &\longrightarrow \mathcal{R} \\ \rho &\longmapsto \sigma_G\circ\tau^{\frac{1}{2}\mp\frac{1}{2}}\circ\rho\circ(\sigma_X)_*\,. \end{aligned} \tag{3.5.19}$$

Definition 3.5.2 The **holonomy representation** $\mathrm{hol}(D) : \pi_1(X,x) \to G$ induced by a flat connection D of a C^∞-bundle $\mathbb{E}$ over X is defined as follows: let $\gamma \in \pi_1(X,x)$, there is a lift $\tilde{\gamma} : [0,1] \to \mathbb{E}$ of γ such that $D(\tilde{\gamma}_*\frac{\partial}{\partial t}) = 0$. Let $e = \tilde{\gamma}(0)$. If $\tilde{\gamma}(1) = eg$, then $\mathrm{hol}(D)(\gamma) := g$.

Consider the homeomorphism of moduli spaces

$$\begin{aligned} \mathrm{hol} :\quad \mathcal{M} &\xrightarrow{\cong} \mathcal{R} \\ (E,\varphi) &\longmapsto \mathrm{hol}(D_{(E,\varphi)})\,. \end{aligned}$$

Proposition 3.5.2 The following diagram commutes

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\mathrm{hol}"] \arrow[d, "i_{\mathcal{M}}(\sigma_X,\sigma_G)^\pm"'] & \mathcal{R} \arrow[d, "i_{\mathcal{R}}(\sigma_X,\sigma_G)^\pm"] \\ \mathcal{M} \arrow[r, "\mathrm{hol}"] & \mathcal{R}. \end{tikzcd}$$

Proof: For any $(E,\varphi)\in\mathcal{M}$, by the properties of holonomy

$$\begin{aligned} i_{\mathcal{R}}(\sigma_X,\sigma_G)^+(\mathrm{hol}(D_{(E,\varphi)})) &= \sigma_G\circ \mathrm{hol}(D_{(E,\varphi)})\circ(\sigma_X)_* = \\ = \mathrm{hol}(D_{(\sigma_X^*\sigma_G E,\sigma_X^*\sigma_G\varphi)}) &= \mathrm{hol}(D_{i_{\mathcal{M}}(\sigma_X,\sigma_G)^+(E,\varphi)})\,. \end{aligned} \tag{3.5.20}$$

On the other hand, by Proposition 3.5.1 we have:

$$\begin{aligned} & i_{\mathcal{R}}(\sigma_X,\sigma_G)^-(\mathrm{hol}(D_{(E,\varphi)})) = \sigma_G\circ\tau^{\frac{1}{2}\mp\frac{1}{2}}\circ \mathrm{hol}(D_{(E,\varphi)})\circ(\sigma_X)_* = \\ & = \mathrm{hol}(D_{\sigma_X^*\sigma_G\circ\tau^{\frac{1}{2}\mp\frac{1}{2}}(E,\varphi)}) = \mathrm{hol}(i_{\mathcal{S}}(\sigma_X,\sigma_G)^- D_{(E,\varphi)}) = \\ & = \mathrm{hol}(D_{i_{\mathcal{M}}(\sigma_X,\sigma_G)^-(E,\varphi)})\,. \end{aligned} \tag{3.5.21}$$

□

3.6 Description of the fixed point locus

As a result of Corollary 3.2.2 there is a forgetfull map

$$\begin{array}{rccc} f_{\mathcal{M}}: & \mathcal{M}(\sigma_X,\sigma_G,c,\pm) & \longrightarrow & \mathcal{M} \\ & (E,\varphi,\sigma_E) & \longmapsto & (E,\varphi)\,. \end{array}$$

We denote by $f_{\mathcal{S}}$ and $f_{\mathcal{R}}$ the forgetfull maps

$$\begin{array}{rccc} f_{\mathcal{S}}: & \mathcal{S}(\sigma_X,\sigma_G,c,\pm) & \longrightarrow & \mathcal{S} \\ & (\mathbb{E},\sigma_{\mathbb{E}},D) & \longmapsto & (\mathbb{E},D)\,. \end{array}$$

$$\begin{array}{rccc} f_{\mathcal{R}}: & \mathcal{R}(\sigma_X,\sigma_G,c,\pm) & \longrightarrow & \mathcal{R} \\ & \rho & \longmapsto & \rho|_{\pi_1(X)}\,. \end{array}$$

We denote by $\widetilde{\mathcal{M}}(\sigma_X,\sigma_G,c,\pm)$, $\widetilde{\mathcal{S}}(\sigma_X,\sigma_G,c,\pm)$ and $\widetilde{\mathcal{R}}(\sigma_X,\sigma_G,c,\pm)$ the images of the forgetfull maps $f_{\mathcal{M}}$, $f_{\mathcal{S}}$ and $f_{\mathcal{R}}$, respectively.

Proposition 3.6.1 The fixed points of $i_{\mathcal{M}(\alpha)}(\sigma_X,\sigma_G)^\pm$ and the moduli space of real polystable G-Higgs bundles are related by:

(1)

$$\mathcal{M}^{i_{\mathcal{M}}(\sigma_X,\sigma_G)^\pm} \supseteq \bigcup_{c\in Z(H^{\mathbb{C}})_2^{\sigma_G}\cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X,\sigma_G,c,\pm)\,.$$

(2) For $g(X) \geq 2$ and if we restrict to the smooth locus sm, then

$$\mathcal{M}_{\mathrm{sm}}^{i_{\mathcal{M}}(\sigma_X,\sigma_G)^{\pm}} \cong \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \pm)_{\mathrm{sm}}\,.$$

Proof: It is a consequence of Proposition 2.10.1. □

Proposition 3.6.2 The fixed points of the involution $i_{\mathcal{S}}(\sigma_X, \sigma_G)$ and the moduli space of $(\pm, c)$-compatible local systems are related by:

(1)

$$\mathcal{S}^{i_{\mathcal{S}}(\sigma_X,\sigma_G)^{\pm}} \supseteq \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{S}}(\sigma_X, \sigma_G, c, \pm)\,.$$

(2) For $g(X) \geq 2$ and if we restrict to the smooth locus sm, then

$$\mathcal{S}_{\mathrm{sm}}^{i_{\mathcal{S}}(\sigma_X,\sigma_G)^{\pm}} \cong \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{S}}(\sigma_X, \sigma_G, c, \pm)_{\mathrm{sm}}\,.$$

Proof: It is a consequence of Proposition 3.6.1, Proposition 3.5.1 and the bijection of moduli spaces $\mathcal{M}(\sigma_X, \sigma_G, c, \pm)$ and $\mathcal{S}(\sigma_X, \sigma_G, c, \pm)$ proved in Theorems 3.2.1 and 3.4.5. □

Proposition 3.6.3 The fixed points of the involution $i_{\mathcal{R}}(\sigma_X, \sigma_G)$ and moduli space of (σ_X, σ_G, c)-equivariant representations of $\Gamma(X)$ are related by:

(1)

$$\mathcal{R}^{i_{\mathcal{R}}(\sigma_X,\sigma_G)^{\pm}} \supseteq \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{R}}(\sigma_X, \sigma_G, c, \pm)\,.$$

(2) For $g(X) \geq 2$ and if we restrict to the smooth locus sm, then

$$\mathcal{R}_{\mathrm{sm}}^{i_{\mathcal{R}}(\sigma_X,\sigma_G)^{\pm}} \cong \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{R}}(\sigma_X, \sigma_G, c, \pm)_{\mathrm{sm}}\,.$$

Proof: It is a consequence of Proposition 3.6.1, Proposition 3.5.1 and Theorem 3.4.6. □

Chapter 4

Real Higgs bundles over elliptic curves

Along this chapter, we use the following notation:

- $X-$elliptic curve
- α_-, α_+-antiholomorphic, holomorphic involutions on X, respectively
- $\epsilon-$parameter with values $\{+,-\}$
- $\alpha_\epsilon-$ holomorphic or antiholomorphic involution on X
- $\widehat{X} := \mathrm{Pic}^0(X)$
- $G-$connected complex reductive Lie group.
- σ_-, σ_+-antiholomorphic, holomorphic involutions on G, respectively
- $\sigma_\epsilon-$ holomorphic or antiholomorphic involution on G
- $T-$Cartan subgroup of G
- Λ_T-cocharacter lattice of T
- $W-$Weyl group associated to (T, G).

4.1 Review on Higgs bundles over elliptic curves

Let (X, x_0) be an elliptic curve. By abuse of notation, we refer to the elliptic curve simply as X. Let G be a connected complex reductive Lie group. We denote by $\mathcal{M}(G)$ to the moduli space of G-Higgs bundles over X and $\mathcal{R}(G)$ the moduli space of representations of the fundamental group of X into G. There is a homeomorphism

$$\mathcal{M}(G) \cong \mathcal{R}(G)\,. \tag{4.1.1}$$

Let $T \subset G$ be a Cartan subgroup, and let $\Lambda_T := \mathrm{Hom}(\mathbb{C}^*, T)$ be the corresponding cocharacter lattice. We denote by $Z(G)$ the center of G, and by Z the connected component of it containing the identity element. Fix, once and for all, a orthogonal basis

$$\{\lambda_1, \cdots, \lambda_\ell, \lambda_{\ell+1}, \cdots, \lambda_s\}$$

of Λ_T such that $\{\lambda_1, \cdots, \lambda_\ell\} \oplus \{\lambda_{\ell+1}, \cdots, \lambda_s\}$ is a decomposition of it, with $\{\lambda_1, \cdots, \lambda_\ell\}$ being an orthogonal basis of $\Lambda_Z := \mathrm{Hom}(\mathbb{C}^*, Z)$. We consider the natural isomorphism

$$\begin{array}{rccc} \eta : & \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T & \stackrel{\cong}{\longrightarrow} & T \\ & \sum z_i \otimes_{\mathbb{Z}} \lambda_i & \longmapsto & \prod \lambda_i(z_i). \end{array}$$

Let $\mu : T \times T \longrightarrow T$ be the multiplication map of the group T. For any two principal T-bundles E and E' over X of trivial topological type, consider the principal $(T \times T)$-bundle $E \times_X E'$, and define

$$E \otimes E' := \mu_*(E \times_X E')\,,$$

which is again a principal T-bundle of trivial topological type. Note that $E(\mathfrak{t}) \cong \mathfrak{t} \otimes \mathcal{O}_X$, so Higgs fields on a principal T-bundle are elements of $\mathfrak{t}$. We denote by $\widehat{X} := \mathrm{Pic}^0(X)$ the Picard group of degree 0. One can express the extension of structure groups associated to η as follows

$$\begin{array}{rccc} \eta_* : & T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T & \stackrel{\cong}{\longrightarrow} & \mathcal{M}(T) \\ & \sum (L_i, \psi_i) \otimes \lambda_i & \longmapsto & \left(\bigotimes (\lambda_i)_* L_i \,,\, \sum (d\lambda_i)_* \psi_i \right)\,. \end{array}$$

Consider also the extension of structure group associated to the injection $T \hookrightarrow G$, and compose it with η_*, to obtain the following map

$$\dot{\xi} : T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \longrightarrow \mathcal{M}(G)\,.$$

Analogously, one can define

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T & \xrightarrow{\dot{\zeta}} & \mathcal{R}(G) \\ \sum_i \rho_i \otimes \lambda_i & \longmapsto & \prod_i (\lambda_i \circ \rho_i) \,. \end{array}$$

Furthermore, the diffeomorphism between $T^*\widehat{X}$ and $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ induced by the Hodge correspondence induces the top row map of the commuting diagram

$$\begin{tikzcd} T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\text{diffeo.}"] \arrow[d, "\dot{\xi}"'] & \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \arrow[d, "\dot{\zeta}"] \\ \mathcal{M}(G) \arrow[r, "\text{diffeo.}"] & \mathcal{R}(G) \,. \end{tikzcd} \tag{4.1.2}$$

The action of the Weyl group $W := N_G(T)/T$ associated to (T, G) on Λ_T can be extended to $A \otimes_{\mathbb{Z}} \Lambda_T$, where A is any abelian group, as follows

$$\omega \cdot \left(\sum a_i \otimes \lambda_i \right) := \sum a_i \otimes \omega(\lambda_i) \,,$$

for all $\omega \in W$ and $a_i \in A$. If A is the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, this action commutes with the natural action of W on T,

$$\begin{tikzcd} \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\eta"] \arrow[d, "\omega\cdot"'] & T \arrow[d, "\omega\cdot"] \\ \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\eta"] & T \,. \end{tikzcd} \tag{4.1.3}$$

The commutativity of (4.1.3) implies that $\dot{\xi}$ and $\dot{\zeta}$ factor through the quotient by the action of W. Thaddeus described in [81] the normalization $\overline{\mathcal{M}}(G)$ of $\mathcal{M}(G)$ as

$$\overline{\mathcal{M}}(G) \cong (T^*X \otimes_{\mathbb{Z}} \Lambda_T)/W \,.$$

Theorem 4.1.1 ([34]) There is a reduced subscheme $\mathcal{M}(G)_{\mathrm{red}} \subset \mathcal{M}(G)$ such that the morphism $\dot{\xi}$ induce an isomorphism

$$\xi \,:\, (T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T)/W \xrightarrow{\cong} \mathcal{M}(G)_{\mathrm{red}} \,.$$

We denote by $\mathcal{R}(G)_{\mathrm{red}}$ the moduli space that corresponds to $\mathcal{M}(G)_{red}$ via the homeomorphism (4.1.1). The morphism $\dot{\zeta}$ induce an isomorphism

$$\zeta \,:\, (\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T)/W \xrightarrow{\cong} \mathcal{R}(G)_{\mathrm{red}} \,.$$

By the commutative diagram (4.1.2), one can see that the follow diagram commutes

$$\begin{tikzcd} (T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T)/W \arrow[r, "\text{diffeo.}"] \arrow[d, "\xi"'] & (\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T)/W \arrow[d, "\zeta"] \\ \mathcal{M}(G)_{\mathrm{red}} \arrow[r, "\text{diffeo.}"] & \mathcal{R}(G)_{\mathrm{red}} . \end{tikzcd} \tag{4.1.4}$$

4.2 Involutions of the moduli space

For any holomorphic involution $\alpha_+ : X \longrightarrow X$, we define

$$\begin{array}{rccc} I(\alpha_+, \sigma_+, +): & \mathcal{M}(G) & \longrightarrow & \mathcal{M}(G) \\ & (E, \varphi) & \longmapsto & (\alpha_+^*\sigma_+(E), +\alpha_+^*\sigma_+(\varphi)) , \end{array} \tag{4.2.5}$$

$$\begin{array}{rccc} I(\alpha_+, \sigma_+, -): & \mathcal{M}(G) & \longrightarrow & \mathcal{M}(G) \\ & (E, \varphi) & \longmapsto & (\alpha_+^*\sigma_+(E), -\alpha_+^*\sigma_+(\varphi)) . \end{array} \tag{4.2.6}$$

For an anti-holomorphic involution $\alpha_- : X \longrightarrow X$, we define

$$\begin{array}{rccc} I(\alpha_-, \sigma_-, +): & \mathcal{M}(G) & \longrightarrow & \mathcal{M}(G) \\ & (E, \varphi) & \longmapsto & (\alpha_-^*\sigma_-(E), -\alpha_-^*\sigma_-(\varphi)) , \end{array} \tag{4.2.7}$$

$$\begin{array}{rccc} I(\alpha_-, \sigma_-, -): & \mathcal{M}(G) & \longrightarrow & \mathcal{M}(G) \\ & (E, \varphi) & \longmapsto & (\alpha_-^*\sigma_-(E), -\alpha_-^*\sigma_-(\varphi)) . \end{array} \tag{4.2.8}$$

Example 4.2.1 In the particular case of real vector bundles over a real elliptic curve (X, σ_X) an explicit description of the involutions $I(\alpha_-, \sigma_-, +)$ was done in [19]. Recall that according to its topological type, (X, σ_X) is a Klein bottle `K`, a Moëbius strip `M` or a closed annulus `A` (See Table 1.1). Their result is the following:

Proposition 4.2.1 ([19], Theorem 1.2) Let h be g.c.d. of the pair of numbers $\{r, d\}$. There is an isomorphism of real manifolds

$$(\mathcal{M}_X(r,d), \sigma_{\mathcal{M}_X(r,d)}) \cong \begin{cases} (\mathrm{Sym}^h(X), \sigma_{\mathrm{Sym}^h(X)}) & \text{if } (X, x_o, \sigma_X) \text{ is } \bar{\mathtt{A}} \text{ or } \mathtt{M} \\ (\mathrm{Sym}^h(X), \sigma_{\mathrm{Sym}^h(X)}) & \text{if } (X, x_o, \sigma_X) \text{ is } \mathtt{K} \text{ and } d/h \text{ is odd} \\ (\mathrm{Sym}^h(\widehat{X}), \sigma_{\mathrm{Sym}^h(\widehat{X})}) & \text{if } (X, x_o, \sigma_X) \text{ is } \mathtt{K} \text{ and } d/h \text{ is even,} \end{cases}$$

where the $\mathrm{Sym}^h(X)$ and $\mathrm{Sym}^h(\widehat{X})$ admit $\sigma_{\mathrm{Sym}^h(X)}$ and $\sigma_{\mathrm{Sym}^h(\widehat{X})}$ are the antiholomorhic involutions in the symmetric spaces spaces induced by σ_X.

Sometimes we write $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ when we want to deal with these involutions simultaneously. The parameter ϵ indicates whether it is the holomorphic ($\epsilon = +$) case or the anti-holomorphic ($\epsilon = -$) case.

See [14, 40] for a proof of the following proposition.

Proposition 4.2.2 If $\sigma_\epsilon \sim \sigma'_\epsilon$, then $I(\alpha_\epsilon, \sigma_\epsilon, \pm) = I(\alpha_\epsilon, \sigma'_\epsilon, \pm)$.

Let $\mathcal{M}(G)_{sm} \subset \mathcal{M}(G)$ be the smooth locus. This manifold $\mathcal{M}(G)_{sm}$ is hyper-Kähler with three complex structures $\Gamma_1, \Gamma_3, \Gamma_3$ and three associated Kähler forms $\omega_1, \omega_2, \omega_3$. Also recall that $\mathcal{M}(G)_{sm}$ has three (holomorphic) symplectic structures: $\Omega_1 = \omega_2 + \sqrt{-1}\omega_3$, $\Omega_2 = \omega_3 + \sqrt{-1}\omega_1$ and $\Omega_3 = \omega_1 + \sqrt{-1}\omega_2$. Now, we describe the geometric structure of the fixed points of the involutions in $\mathcal{M}(G)_{sm}$ defined above (see [14, 40] for instance):

- the fixed point locus of $I(\alpha_+, \sigma_+, +)$ is a hyper-Kähler submanifold,
- the fixed point locus of $I(\alpha_+, \sigma_+, -)$ is a Ω_1-Lagrangian submanifold,
- the fixed point locus of $I(\alpha_-, \sigma_-, -)$ is a Ω_2-Lagrangian submanifold,
- the fixed point locus of $I(\alpha_-, \sigma_-, +)$ is a Ω_3-Lagrangian submanifold.

Remark 4.2.1 In the context of Mirror Symmetry:

- the fixed point locus of $I(\alpha_+, \sigma_+, +)$ is a (B, B, B)-brane,
- the fixed point locus of $I(\alpha_+, \sigma_+, -)$ is a (B, A, A)-brane,
- the fixed point locus of $I(\alpha_-, \sigma_-, -)$ is a (A, B, A)-brane,
- the fixed point locus of $I(\alpha_-, \sigma_-, +)$ is a (A, A, B)-brane.

The above involutions induce the following involutions of $\mathcal{R}(G)$

$$\begin{array}{rccc} J(\alpha_+, \sigma_+, +): & \mathcal{R}(G) & \longrightarrow & \mathcal{R}(G) \\ & \rho & \longmapsto & \sigma_+ \circ \rho \circ (\alpha_+)_* \ , \end{array} \tag{4.2.9}$$

$$\begin{array}{rccc} J(\alpha_-, \sigma_-, +): & \mathcal{R}(G) & \longrightarrow & \mathcal{R}(G) \\ & \rho & \longmapsto & \sigma_- \circ \rho \circ (\alpha_-)_* \ , \end{array} \tag{4.2.10}$$

$$
\begin{aligned}
J(\alpha_+, \sigma_+, -) : \quad \mathcal{R}(G) & \longrightarrow \quad \mathcal{R}(G) \\
\rho & \longmapsto \sigma_- \circ \rho \circ (\alpha_+)_* \,,
\end{aligned}
\tag{4.2.11}
$$

$$
\begin{aligned}
J(\alpha_-, \sigma_-, -) : \quad \mathcal{R}(G) & \longrightarrow \quad \mathcal{R}(G) \\
\rho & \longmapsto \sigma_+ \circ \rho \circ (\alpha_-)_* \,.
\end{aligned}
\tag{4.2.12}
$$

Let $Z = Z(G)_0$ be the connected component of the center $Z(G) \subset G$ containing the identity element. Define the homomorphism

$$
\begin{aligned}
\mu : \quad Z \times G & \longrightarrow \quad G \\
(y\,, z) & \longmapsto \quad yz\,.
\end{aligned}
$$

For any principal G-bundle E and any principal Z-bundle F, we define the G-bundle

$$
F \otimes E \,:=\, \mu_*(F \times_X E) \;;
$$

note that the fiber product $F \times_X E$ is a principal $(Z \times G)$-bundle and hence $\mu_*(F \times_X E)$ is a G-bundle. It is straight-forward to check that $F \otimes E$ is semistable, stable or polystable if and only if E is semistable, stable or polystable respectively. If F' is a Z-bundle, then using the multiplication operation $\mu' : Z \times Z \longrightarrow Z$ we get a Z-bundle

$$
F \otimes F' \,:=\, \mu'_*(F \times_X F') \;.
$$

This operation makes the moduli space $M(Z)$ of topologically trivial Z-bundles on X a complex Lie group. The Lie algebra of Z will be denoted by $\mathfrak{z}$. The adjoint bundle $F(\mathfrak{z})$ for a principal Z-bundle F is the trivial vector bundle $\mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{z}$ over X with fiber $\mathfrak{z}$. The Lie algebra of G will be denoted by $\mathfrak{g}$. For the adjoint action of G on $\mathfrak{g}$, each point of $\mathfrak{z}$ is fixed. Hence

$$
H^0(X,\, F(\mathfrak{z})) \,=\, \mathfrak{z} \,\subset\, H^0(X,\, E(\mathfrak{g})) \;.
$$

We denote by $\mathcal{M}(Z)$ the moduli space of Higgs Z-bundles on X. $\mathcal{M}(Z)$ is a group and it has the following holomorphic action on $\mathcal{M}(G)$

$$
\begin{aligned}
\mathcal{M}(Z) \times \mathcal{M}(G) & \longrightarrow \qquad \mathcal{M}(G) \\
((F,\, \phi),\, (E,\, \varphi)) & \longmapsto \quad (F,\, \phi) \otimes (E,\, \varphi) = (F \otimes E,\, \phi + \varphi)\,.
\end{aligned}
\tag{4.2.13}
$$

Analogously, given a homomorphism

$$
\chi \,:\, \pi_1(X) \longrightarrow Z\,,
$$

one gets for any homomorphism $\rho : \pi_1(X) \longrightarrow G$ a homomorphism $\chi \cdot \rho : \pi_1(X) \longrightarrow G$ given by the composition

$$\pi_1(X) \xrightarrow{\chi \times \rho} Z \times G \xrightarrow{\mu} G\,.$$

More precisely, $\mathcal{R}(Z)$ is a complex Lie group that acts holomorphically on $\mathcal{R}(G)$ as follows

$$\begin{array}{rcl} \mathcal{R}(Z) \times \mathcal{R}(G) & \longrightarrow & \mathcal{R}(G) \\ (\chi,\, \rho) & \longmapsto & \chi \cdot \rho\,. \end{array} \tag{4.2.14}$$

Since any anti-holomorphic involution σ_- of G preserves $Z \subset G$, we can combine (4.2.13) and (4.2.14) to obtain more involutions. For every $\mathcal{F} = (F,\, \phi) \in \mathcal{M}(Z)$, we define

$$\begin{array}{rrcl} I(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F}) : & \mathcal{M}(G) & \longrightarrow & \mathcal{M}(G) \\ & (E, \varphi) & \longmapsto & \mathcal{F} \otimes I(\alpha_\epsilon, \sigma_\epsilon, \pm)(E, \varphi)\,. \end{array}$$

It is an involution whenever $\mathcal{F} = (F,\, \phi)$ satisfies

$$\mathcal{F}^{-1} := (F^{-1},\, -\phi) = I(\alpha_\epsilon, \sigma_\epsilon, \pm)(F, \phi)\,, \tag{4.2.15}$$

where F^{-1} coincides with the Z-bundle obtained by extending the structure group of F using the automorphism $z \longmapsto z^{-1}$ of Z.

Analogously, we denote by $\mathcal{R}(Z)$ the moduli space of representations of the fundamental group of X into Z. For every $\chi \in \mathcal{R}(Z)$, define

$$\begin{array}{rrcl} J(\alpha_\epsilon, \sigma_\epsilon, \pm, \chi)\,. : & \mathcal{R}(G) & \longrightarrow & \mathcal{R}(G) \\ & \rho & \longmapsto & \chi \cdot J(\alpha_\epsilon, \sigma_\epsilon, \pm)(\rho) \end{array}$$

The condition for $J_{\alpha_\epsilon}^{\sigma_\epsilon, \chi}$ to be an involution is that

$$\chi^{-1} = J(\alpha_\epsilon, \sigma_\epsilon, \pm)(\chi)\,. \tag{4.2.16}$$

Theorem 4.2.1 For all $\mathcal{F} \in \mathcal{M}(Z)$ and $\chi \in \mathcal{R}(Z)$ satisfying (4.2.15) and (4.2.16), the above maps $I(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F})$ and $J(\alpha_\epsilon, \sigma_\epsilon, \pm, \chi)$ are involutions.

The fixed points of $I(\alpha_+, \sigma_+, +, \mathcal{F})$ restricted to $\mathcal{M}(G)_{sm}$ are (B, B, B)-branes, while the fixed points of $I(\alpha_+, \sigma_+, -, \mathcal{F})$ are (B, A, A)-branes. On the other hand, the fixed points of $I(\alpha_-, \sigma_-, -, \mathcal{F})$ are (A, B, A)-branes when we restrict to $\mathcal{M}(G)_{sm}$ and the fixed points of $I(\alpha_-, \sigma_-, +, \mathcal{F})$ are (A, A, B)-branes.

If $\mathcal{F} = (F, \phi) \in \mathcal{M}(Z)$ is the Z-Higgs bundle associated to the representation $\chi \in \mathcal{R}(Z)$, then the diagram

$$\begin{tikzcd} \mathcal{M}(G) \arrow[r, "\text{homeo.}"] \arrow[d, "I(\alpha_\epsilon,\sigma_\epsilon,\pm,\mathcal{F})"'] & \mathcal{R}(G) \arrow[d, "J(\alpha_\epsilon,\sigma_\epsilon,\pm,\chi)"] \\ \mathcal{M}(G) \arrow[r, "\text{homeo.}"] & \mathcal{R}(G) \end{tikzcd} \tag{4.2.17}$$

is commutative.

Proof: The statements for $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ follow from the works [19], [14] and [40]. One can prove that the diffeomorphism in the correspondence that takes the map in (4.2.13) to the map in (4.2.14). Then $I(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F})$ commutes or anticommutes with Γ_1 and Γ_2 whenever $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ does so. By the above references, the involutions $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ preserve the hyper-Kähler metric on $\mathcal{M}(G)_{sm}$. Note that the translation (4.2.13) preserves the hyper-Kähler metric as well, and so does $I(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F})$. As a consequence, the statements proved for $I(\alpha_\epsilon, \sigma_\epsilon, \pm)$ are also valid for $I(\alpha_\epsilon, \sigma_\epsilon, \pm, \mathcal{F})$. $\square$

Remark 4.2.2 When $\sigma_+ = \mathrm{Id}_G$, the involution $I(\alpha_+, \mathrm{Id}_G, +)$ is just the pull-back by α_+, while $J(\alpha_+, \mathrm{Id}_G, +)$ coincides with the composition of homomorphisms with $(\alpha_+)_*$. Denote by Id_X^{-1} and Id_G^{-1} the inversion given by the group structures on X and G, respectively. For any representation of the fundamental group, note that $\rho \circ (\mathrm{Id}_X^{-1})_* = \rho^{-1} = \mathrm{Id}_G^{-1} \circ \rho$. As a consequence of the commutativity of (4.2.17), the pull-back by Id_X^{-1} commutes with the extension of structure group associated to Id_G^{-1}.

Along the chapter, we use the same notations that in Section 1.1.1 for holomophic or antiholomorphic involutions $\alpha_{(\epsilon,a)}$ on X and for translations t_y.

Remark 4.2.3 Every semistable principal G-bundle $E \longrightarrow X$ of trivial topological type is homogeneous, meaning

$$t_y^* E \cong E$$

for every $y \in X$ [**?**, Theorem 4.1]. This implies that the pull-back by $\alpha_{(+,a)}$ coincides with the pull-back by $t_y \circ \alpha_{(+,a)}$; therefore, both involutions define the same involution on the moduli space of G–bundles of trivial topological type, meaning

$$I(t_y \circ \alpha_{(+,a)}, \sigma_+, \pm) = I(\alpha_{(+,a)}, \sigma_+, \pm).$$

Since the fundamental group of X is abelian, $\pi_1(X, x_0)$ is identified with $\pi_1(X, y)$ (in general they are identified uniquely up to an inner automorphism); with this identification, the action of $(t_y)_*$ on $\pi_1(X)$ is trivial. Therefore,

$$J(t_y \circ \alpha_{(+,a)}, \sigma_+, \pm) = J(\alpha_{(+,a)}, \sigma_+, \pm).$$

Remark 4.2.4 As in the previous Remark 4.2.3, the homogeneity of topologically trivial semistable principal G-bundles on elliptic curves implies that the pullbacks by $\alpha_{(-,a)}$ and $t_y \circ \alpha_{(-,a)}$ of such a bundle are isomorphic. We also have the identifications

$$I(t_y \circ \alpha_{(-,a)},\, \sigma_-, \pm) \;=\; I(\alpha_{(-,a)},\, \sigma_-, \pm)$$

and

$$J(t_y \circ \alpha_{(-,a)},\, \sigma_-, \pm) \;=\; J(\alpha_{(-,a)},\, \sigma_-, \pm)$$

in the anti-holomorphic case. The notation used here is guided by these identities.

4.3 Description of the involutions in the rank 1 case

The identity map of the multiplicative group $\mathbb{C}^*$ is associated to the anti-holomorphic involution of the compact real form $\mathrm{U}(1)$

$$\begin{array}{rccc} \sigma_{\mathrm{U}(1)}: & \mathbb{C}^* & \longrightarrow & \mathbb{C}^* \\ & z & \longmapsto & \overline{z}^{-1}. \end{array} \tag{4.3.18}$$

For the group $G = \mathbb{C}^*$, let $i(\alpha_{(-,a)}, \pm) \;=\; I(\alpha_{(-,a)}, \sigma_{\mathrm{U}(1)}, \pm)$. Assume that $\mathcal{F} = (F,\, \phi)$ satisfies the involution condition in (4.2.15), then we can define the following involutions on $T^*\widehat{X}$

$$\begin{array}{rccc} i(\alpha_{(-,a)}, \pm, \mathcal{F}): & T^*\widehat{X} & \longrightarrow & T^*\widehat{X} \\ & (L,\, \psi) & \longmapsto & (F \otimes \alpha_{(-,a)}^* \overline{L}^*,\, \phi \mp \alpha_{(-,a)}^* \overline{\psi}). \end{array}$$

We have that $j(\alpha_{(-,a)}, \pm) \;=\; J(\alpha_{(-,a)}, \sigma_{\mathrm{U}(1)}, \pm)$. Similarly, if χ satisfies (4.2.16), we can consider the involution on $\mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)$

$$
\begin{array}{rccc}
j(\alpha_{(\epsilon,a)}, -, \chi): & \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) & \longrightarrow & \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \\
& \rho & \longmapsto & \chi \cdot \left(\overline{\rho}^{-1} \circ \left(\alpha_{(\epsilon,a)}\right)_*\right) .
\end{array}
$$

Since $T^*\widehat{X} \cong \widehat{X} \times H^0(X, \mathcal{O}_X)$, the isomorphism p_{x_0} in (1.1.4) produces an isomorphism

$$
p_{x_0} : X \times H^0(X, \mathcal{O}_X) \xrightarrow{\cong} T^*\widehat{X} .
$$

Given an anti-holomorphic involution, $\alpha : X \longrightarrow X$, let

$$
\begin{array}{rccc}
\mathrm{conj}: & H^0(X, \mathcal{O}_X) & \longrightarrow & H^0(X, \mathcal{O}_X) \\
& X & \longmapsto & \alpha^*\overline{X}
\end{array}
$$

be the induced conjugate-linear homomorphism.

Lemma 4.3.1 Take $\mathcal{F} = (F, \phi)$ satisfying (4.2.15), and let $y \in X$ be such that $F = p_{x_0}(y)$. Then,

$$
i(\alpha_{(-,a)}, \pm, \mathcal{F}) = p \circ (t_y \circ \alpha_{(-,-a)}, \mp\mathrm{conj}) \circ p^{-1} .
$$

Proposition 4.3.1 The first statement is straight-forward. The lemma follows from the fact that for all $x \in X_\gamma$,

$$
\begin{aligned}
\alpha^*_{(-,a)}\sigma_{\mathrm{U}(1)}\left(\mathcal{O}_X(x - x_0)\right) &= \alpha^*_{(-,a)}\overline{\mathcal{O}(x - x_0)}^* \\
&= \mathcal{O}_X(-\alpha_{(-,a)}(x) - x_0) \\
&= \mathcal{O}_X(\alpha_{(-,-a)}(x) - x_0) .
\end{aligned}
$$

Recall the generators δ_1, δ_2 of $\pi_1(X)$ defined in (1.1.5) and (1.1.6). One has the isomorphism

$$
\begin{array}{rccc}
q: & \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) & \xrightarrow{\cong} & \mathbb{C}^* \times \mathbb{C}^* \\
& \rho & \longmapsto & (\rho(\delta_1), \rho(\delta_2))
\end{array}
$$

Remark 4.3.1 Note that the involution $j(\alpha_{(-,a)}, \pm, \chi)$ is not defined for every isomorphism class of elliptic curves; it is defined only for certain values specified in Table 1.1.

For each region R of Table 1.1, or the entire upper-half plane $\mathbb{H}$ when $\epsilon = 1$, we consider the automorphism

$$
f^{\pm}_{(\epsilon,a,R)} := q \circ j(\alpha_{(\epsilon,a)}, \pm) \circ q^{-1}
$$

of $\mathbb{C}^* \times \mathbb{C}^*$.

Remark 4.3.2 Given $\chi \in \mathcal{R}(Z)$, set $b_i := \chi(\delta_i)$. Note that χ satisfies (4.2.16) if and only if

$$(b_1^{-1}, b_2^{-1}) = f^{\pm}_{(\epsilon,a,R)}(b_1, b_2) \ .$$

If this holds, then

$$\begin{array}{rccc} f^{\pm,(b_1,b_2)}_{(\epsilon,a,R)} : & \mathbb{C}^* \times \mathbb{C}^* & \longrightarrow & \mathbb{C}^* \times \mathbb{C}^* \\ & (z_1,\, z_2) & \longmapsto & (b_1, b_2) \cdot f^{\pm}_{(\epsilon,a,R)}(z_1,\, z_2) \end{array}$$

are involutions, and

$$j(\alpha_{(\epsilon,a)}, \pm, \chi) = q^{-1} \circ f^{\pm,(b_1,b_2)}_{(\epsilon,a,R)} \circ q \ .$$

Using Tables 1.5 and 1.3, we describe $f^{\pm}_{(\epsilon,a,R)}$, and therefore $f^{\pm,(b_1,b_2)}_{(\epsilon,a,R)}$, for any involution $\alpha_{(\epsilon,a)}$ of the curve in the following Table 4.1.

Region	Involution	$f^{+}_{(\epsilon,a,R)}(z_1, z_2)$	$f^{-}_{(\epsilon,a,R)}(z_1, z_2)$
$\mathcal{H}$	$\alpha_{(+,1)}$	(z_1, z_2)	$(\bar{z}_1^{-1}, \bar{z}_2^{-1})$
	$\alpha_{(+,-1)}$	(z_1^{-1}, z_2^{-1})	$(\bar{z}_1, \bar{z}_2)$
A, B	$\alpha_{(-,1)}$	(z_1, z_2^{-1})	$(\bar{z}_1^{-1}, \bar{z}_2)$
	$\alpha_{(-,-1)}$	(z_1^{-1}, z_2)	$(\bar{z}_1, \bar{z}_2^{-1})$
C, D	$\alpha_{(-,1)}$	(z_1, z_2^{-1})	$(\bar{z}_1^{-1}, \bar{z}_2)$
	$\alpha_{(-,-1)}$	(z_1^{-1}, z_2)	$(\bar{z}_1, \bar{z}_2^{-1})$
	$\alpha_{(-,\gamma)}$	(z_2, z_1)	$(\bar{z}_2^{-1}, \bar{z}_1^{-1})$
	$\alpha_{(-,-\gamma)}$	(z_2^{-1}, z_1^{-1})	$(\bar{z}_2, \bar{z}_1)$
E	$\alpha_{(-,1)}$	$(z_1, z_2^{-1} z_1)$	$(\bar{z}_1^{-1}, \bar{z}_2 \bar{z}_1^{-1})$
	$\alpha_{(-,-1)}$	$(z_1^{-1}, z_2 z_1^{-1})$	$(\bar{z}_1^{-1}, \bar{z}_2^{-1} \bar{z}_1)$

Table 4.1: Values of $f^{\pm}_{(\epsilon,a,R)}$.

4.4 Description of the involutions in the general case

Let σ_+ and σ_- respectively be the Cartan (holomorphic) involution and the associated real form (anti-holomorphic involution) of G. Recall from Theorem 1.3.3 that they are related through the composition by the compact real form involution τ commuting with σ_-,

$$\sigma_+ = \sigma_- \tau . \tag{4.4.19}$$

Let T be a Cartan subgroup of G preserved by σ_+, σ_- and τ. (The existence is proved in [23, Theorem 1 p. 329]). We denote by the same symbols the involutions in the cocharacter lattice $\Lambda_T = \mathrm{Hom}(\mathbb{C}^*, T)$,

$$\begin{array}{rccc} \sigma_+ : & \Lambda_T & \longrightarrow & \Lambda_T \\ & \lambda & \longmapsto & \sigma_+ \circ \lambda , \end{array} \tag{4.4.20}$$

and

$$\begin{array}{rccc} \sigma_- : & \Lambda_T & \longrightarrow & \Lambda_T \\ & \lambda & \longmapsto & \sigma_- \circ \lambda \circ \sigma_{\mathrm{U}(1)} . \end{array} \tag{4.4.21}$$

Define an action of τ on Λ_T by $\tau(\lambda) = \tau \circ \lambda \circ \sigma_{\mathrm{U}(1)}$; note that the equality (4.4.19), considered as an equality of involutions of Λ_T, is recovered.

Since the Cartan subgroup T is preserved by σ_+, σ_- and τ, these involutions induce involutions of the normalizer $N_G(T)$, and therefore produce involutions of the Weyl group $W = N_G(T)/Z_G(T)$; these involutions of the Weyl group are denoted by the same symbol.

Remark 4.4.1 Let $T_0 := T^\tau$ be the compact torus fixed pointwise by τ. Since $T_0^{\mathbb{C}} = T$, it follows that $\mathrm{Hom}(\mathbb{C}^*, T) \cong \mathrm{Hom}(\mathrm{U}(1), T_0)$. The action of τ is trivial on $\Lambda_T \cong \mathrm{Hom}(\mathrm{U}(1), T_0)$. Also, we recall that τ acts trivially on W, because it is a compact real form. As a consequence, the actions of σ_+ and σ_- on Λ_T coincide. The same statement holds for the actions of σ_+ and σ_- on W.

As done in (4.4.20) and (4.4.21), define the holomorphic involution

$$\begin{array}{rccc} \dot{\sigma}_+ : & \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T & \longrightarrow & \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \\ & \sum z_i \otimes_{\mathbb{Z}} \lambda_i & \longmapsto & \sum z_i \otimes_{\mathbb{Z}} \sigma_+(\lambda_i) , \end{array}$$

and the anti-holomorphic involution

$$\begin{array}{rccc} \dot{\sigma}_- : & \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T & \longrightarrow & \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \\ & \sum z_i \otimes_{\mathbb{Z}} \lambda_i & \longmapsto & \sum \sigma_{\mathrm{U}(1)}(z_i) \otimes_{\mathbb{Z}} \sigma_-(\lambda_i) . \end{array}$$

We use again $\dot{\sigma}_\epsilon$ to refer simultaneously $\dot{\sigma}_+$ and $\dot{\sigma}_-$. Now the following diagram is commutative

$$\begin{tikzcd} \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\eta"] \arrow[d, "\dot{\sigma}_\epsilon"'] & T \arrow[d, "\sigma_\epsilon"] \\ \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\eta"] & T\,. \end{tikzcd} \tag{4.4.22}$$

The action of W on Λ_T is σ_ϵ-equivariant, meaning for any $\omega \in W$ and any $\lambda \in \Lambda_T$,

$$\sigma_\epsilon(\omega \cdot \lambda) \;=\; \sigma_\epsilon(\omega) \cdot \sigma_\epsilon(\lambda)\,. \tag{4.4.23}$$

Therefore, we have

$$\dot{\sigma}_\epsilon \circ \omega \;=\; \sigma_\epsilon(\omega) \circ \dot{\sigma}_\epsilon\,. \tag{4.4.24}$$

Now define the involutions

$$\begin{aligned} \dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm): \quad T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \quad &\longrightarrow \quad T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \\ \textstyle\sum (L_i, \psi_i) \otimes \lambda_i \quad &\longmapsto \quad \textstyle\sum i(\alpha_{(\epsilon,a)}, \pm)(L_i, \psi_i) \otimes \sigma_\epsilon(\lambda_i)\,, \end{aligned}$$

and

$$\begin{aligned} \dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm): \quad \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \quad &\longrightarrow \quad \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \\ \textstyle\sum \rho_i \otimes \lambda_i \quad &\longmapsto \quad \textstyle\sum j(\alpha_{(\epsilon,a)}, \pm)(\rho_i) \otimes \sigma_\epsilon(\lambda_i)\,. \end{aligned}$$

Fix any $\mathcal{F} = (F, \phi) \in \mathcal{M}(Z)$; let

$$\chi \,:\, \pi_1(X) \longrightarrow Z$$

be the corresponding representation of the fundamental group. We recall that

$$\mathcal{M}(Z) \;\cong\; T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_Z\,, \quad \mathcal{R}(Z) \;\cong\; \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_Z\,.$$

Let $[\mathcal{F}] = \dot{\xi}^{-1}(\mathcal{F})$ and $[\chi] = \dot{\zeta}^{-1}(\chi)$ be the corresponding elements of the groups $T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T$ and $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T$ respectively. Following the definitions of $I(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$ and $J(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$, define

$$\begin{aligned} \dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F}): \quad T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \quad &\longrightarrow \quad T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \\ \textstyle\sum (L_i, \psi_i) \otimes \lambda_i \quad &\longmapsto \quad [\mathcal{F}] + \dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm)\left(\textstyle\sum z_i \otimes_{\mathbb{Z}} \lambda_i\right)\,, \end{aligned}$$

and

$$\begin{aligned} \dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \chi): \quad \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \quad &\longrightarrow \quad \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \\ \textstyle\sum \rho_i \otimes \lambda_i \quad &\longmapsto \quad [\chi] + \dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm)\left(\textstyle\sum \rho_i \otimes \lambda_i\right)\,. \end{aligned}$$

Lemma 4.4.1 The diagrams

$$\begin{tikzcd} T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\dot{\xi}"] \arrow[d, "\dot{I}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})"'] & \mathcal{M}(G) \arrow[d, "I(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})"] \\ T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\dot{\xi}"] & \mathcal{M}(G) \end{tikzcd} \tag{4.4.25}$$

and

$$\begin{tikzcd} \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\dot{\zeta}"] \arrow[d, "\dot{J}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})"'] & \mathcal{R}(G) \arrow[d, "J(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})"] \\ \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "\dot{\zeta}"] & \mathcal{R}(G) \end{tikzcd} \tag{4.4.26}$$

commute.

Proof: In view of (4.4.24) it follows that $\dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$ induces an involution on the quotient $T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T/W$. From (4.4.22) and the construction of ξ it is clear that this induced involution coincides with $I(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$.

The proof of the commutativity of (4.4.26) is analogous. □

In Section 4.1 we chose a basis of Λ_T with an orthogonal decomposition

$$\{\lambda_1, \cdots, \lambda_\ell\} \oplus \{\lambda_{\ell+1}, \cdots, \lambda_s\},$$

where $\{\lambda_1, \cdots, \lambda_\ell\}$ is an orthogonal basis of $\Lambda_Z = \mathrm{Hom}(\mathbb{C}^*, Z)$. Take a Z-Higgs bundle $\mathcal{F} = (F, \phi)$, and set $\mathcal{F}_i^\epsilon = (F_i^\epsilon, \phi_i^\epsilon)$ to be the Higgs line bundles such that

$$[\mathcal{F}] = \sum [\mathcal{F}_i^\epsilon] \otimes \sigma_\epsilon(\lambda_i). \tag{4.4.27}$$

Let us extend this by setting $\mathcal{F}_i^\epsilon := (\mathcal{O}_X, 0)$ for $\ell + 1 \leq i \leq s$. Analogously, we take a decomposition of the representation $\chi : \pi_1(X) \longrightarrow Z$

$$[\chi] = \sum [\chi_i^\epsilon] \otimes \sigma_\epsilon(\lambda_i) \tag{4.4.28}$$

where χ_i^ϵ are homomorphisms from $\pi_1(X)$ to $\mathbb{C}^*$. As before, set $\chi_i^\epsilon = \mathrm{Id}$ for $\ell+1 \leq i \leq s$.

Proposition 4.4.1 For $\mathcal{F}$ and χ as above, construct $\mathcal{F}_i^\epsilon$ and χ_i^ϵ as in (4.4.27) and (4.4.28) respectively. Then

$$\dot{I}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F}) \left(\sum (L_i, \psi_i) \otimes \lambda_i\right) = \sum i(\alpha_{(\epsilon,a)}, \pm, \mathcal{F}_i^\epsilon)(L_i, \psi_i) \otimes \sigma_\epsilon(\lambda_i),$$

and

$$\dot{J}(\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F}) \left(\sum \rho_i \otimes \lambda_i\right) = \sum j(\alpha_{(\epsilon,a)}, \pm, \chi_i^\epsilon)(\rho_i) \otimes \sigma_\epsilon(\lambda_i).$$

Proof: Once we have the descriptions of $\dot{I}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm)$ and $\dot{J}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm)$, it is immediate to derive descriptions of $\dot{I}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})$ and $\dot{J}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\chi)$. We have

$$\begin{aligned}
\dot{I}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})\left(\sum(L_i,\psi_i)\otimes\lambda_i\right) &= \sum[\mathcal{F}_i^\epsilon]\otimes\sigma_\epsilon(\lambda_i)+\sum i(\alpha_{(\epsilon,a)},\pm)(L_i,\psi_i)\otimes\sigma_\epsilon(\lambda_i)\\
&= \sum\left([\mathcal{F}_i^\pm]+i(\alpha_{(\epsilon,a)},\pm)(L_i,\psi_i)\right)\otimes\sigma_\epsilon(\lambda_i)\\
&= \sum i(\alpha_{(\epsilon,a)},\pm,\mathcal{F}_i^\epsilon)(L_i,\psi_i)\otimes\sigma_\epsilon(\lambda_i)\,.
\end{aligned}$$

The case of $\dot{J}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\chi)$ is analogous. □

4.5 Description of the fixed point locus

Consider the cocharacter lattice Λ_T of a Cartan subgroup T of a complex reductive affine algebraic group G. Denote by σ_- the anti-holomorphic involution of a certain real form, and denote by σ_+ the Cartan involution associated to it. Recall from (4.4.23) that the Weyl group W acts σ_ϵ-equivariantly on Λ_T.

Let A be a complex abelian group; construct the tensor product

$$\dot{B} := A\otimes_{\mathbb{Z}}\Lambda_T\,.$$

Consider also the action of W on $\dot{B}$ induced by the action of W on Λ and take its quotient

$$B := \dot{B}/W\,.$$

Fix a basis $\{\lambda_1,\cdots,\lambda_s\}$ of Λ and suppose that A is equipped with a set of analytic group automorphisms of order two $\{t_1,\,\cdots,\,t_s\}$

$$t_i : A\longrightarrow A$$

such that we can combine them with σ_ϵ to obtain the analytic involution

$$\begin{aligned}
\dot{\tau}:\quad \dot{B}=A\otimes_{\mathbb{Z}}\Lambda &\longrightarrow \dot{B}\\
\sum a_i\otimes\lambda_i &\longmapsto \sum t_i(a_i)\otimes\sigma_\epsilon(\lambda_i)\,.
\end{aligned}$$

Since the action of the Weyl group is σ_ϵ-equivariant, it follows that $\dot{\tau}$ induces an involution on the quotient

$$\begin{aligned}
\tau:\quad B=(A\otimes_{\mathbb{Z}}\Lambda)/W &\longrightarrow B\\
[\sum a_i\otimes\lambda_i]_W &\longmapsto [\sum t_i(a_i)\otimes\sigma_\epsilon(\lambda_i)]_W\,.
\end{aligned}$$

Evidently, the diagram

$$\begin{tikzcd} A \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "p"] \arrow[d, "\dot{\tau}"'] & A \otimes_{\mathbb{Z}} \Lambda_T/W \arrow[d, "\tau"] \\ A \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "p"] & A \otimes_{\mathbb{Z}} \Lambda_T/W \end{tikzcd}$$

commutes, where p is the projection induced by the quotient map for the action of W. By construction, we have a similar commuting diagram for every element $\omega \in W$,

$$\begin{tikzcd} A \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "p"] \arrow[d, "\omega\dot{\tau}"'] & A \otimes_{\mathbb{Z}} \Lambda_T/W \arrow[d, "\tau"] \\ A \otimes_{\mathbb{Z}} \Lambda_T \arrow[r, "p"] & A \otimes_{\mathbb{Z}} \Lambda_T/W \end{tikzcd} \tag{4.5.29}$$

The aim of this section is to describe the fixed point set $(A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau$. To do so, we will make use of the commutativity of (4.5.29). Define, for any $\omega \in W$, the subgroup of $A \otimes_{\mathbb{Z}} \Lambda_T/W$

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_\omega := p((A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}}).$$

One has that

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau = \bigcup_{\omega \in W} (A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_\omega .$$

Remark 4.5.1 Note that each $\dot{B}^{\omega\dot{\tau}}$ is a closed subset, because $\omega\dot{\tau}$ acts continuously on $\dot{B}$. The projection p is closed, therefore $B^\tau_\omega = p(\dot{B}^{\omega\dot{\tau}})$ is closed as well.

Lemma 4.5.1 The equality

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_{\omega_1} = (A \otimes_{\mathbb{Z}} \Lambda_T/W)^\tau_{\omega_2}$$

holds if and only if there exists an element $\omega' \in W$ such that

$$\omega_2 = \omega'\omega_1\sigma_\epsilon(\omega')^{-1} . \tag{4.5.30}$$

Proof: We have that $x' \in (A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}}$ if and only if $x' = \omega\dot{\tau}(x')$. Now $x = \omega' x'$ lies in $\omega' \cdot (A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}}$ if and only if

$$(\omega')^{-1}x = \omega\dot{\tau}((\omega')^{-1}x) .$$

In that case,

$$x = \omega'\omega\sigma_\epsilon(\omega')^{-1}\dot{\tau}(x) .$$

We see that for $\omega, \omega' \in W$,

$$\omega' \cdot (A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega \dot{\tau}} = (A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega' \omega \sigma_\epsilon (\omega')^{-1} \dot{\tau}},$$

which implies the lemma. $\square$

Following (4.5.30), define the left σ_ϵ-adjoint action

$$\begin{array}{rrcl} \mathrm{ad}_{\sigma_\epsilon} : & W \times W & \longrightarrow & W \\ & (\omega', \omega) & \longmapsto & \omega' \omega \sigma_\epsilon (\omega')^{-1}. \end{array}$$

From Lemma 4.5.1, it follows that the components are parametrized by

$$W/_{\sigma_\epsilon} W = W/\mathrm{ad}_{\sigma_\epsilon}(W).$$

Therefore, we write

$$(A \otimes_{\mathbb{Z}} \Lambda_T / W)^\tau = \bigcup_{\overline{\omega} \in W/_{\sigma_\epsilon} W} (A \otimes_{\mathbb{Z}} \Lambda_T / W)^\tau_\omega,$$

where ω is a representative of the σ_ϵ-conjugacy class $\overline{\omega} \in W/_{\sigma_\epsilon} W$.

Proposition 4.5.1 Denote by $T^{\omega \sigma_\epsilon}$ the subtorus fixed by $\omega \sigma_\epsilon$. Then

$$(A \otimes_{\mathbb{Z}} \Lambda_T / W)^\tau_\omega \cong ((A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega \dot{\tau}})/N_W(T^{\omega \sigma_\epsilon}). \tag{4.5.31}$$

Furthermore,

$$\dim(B^\tau_\omega) = \dim(A \otimes_{\mathbb{Z}} \Lambda_T) - 1. \tag{4.5.32}$$

Proof: We claim that

$$N_W(T^{\omega \sigma_\epsilon}) = \{\omega' \in W \mid \omega = \omega' \omega \sigma_\epsilon (\omega')^{-1}\}. \tag{4.5.33}$$

To prove this claim, first recall that

$$N_W(T^{\omega \sigma_\epsilon}) = \{\omega' \in W \mid \omega'(T^{\omega \sigma_\epsilon}) = T^{\omega \sigma_\epsilon}\}.$$

Now note that

$$\omega'(T^{\omega \sigma_\epsilon}) = T^{\omega' \omega \circ \sigma_\epsilon \circ (\omega')^{-1}} = T^{\omega' \omega \sigma_\epsilon (\omega')^{-1} \circ \sigma_\epsilon}.$$

We have $T^{\omega' \omega \sigma_\epsilon (\omega')^{-1} \circ \sigma_\epsilon} = T^{\omega \sigma_\epsilon}$ if and only if $\omega' \omega \sigma_\epsilon (\omega')^{-1} = \omega$. This proved the claim.

The first statement in the proposition follows from the combination of Lemma 4.5.1 and the above claim.

Now, we prove the second statement. Since p is the quotient by a finite subgroup, one has

$$\dim(B^{\tau}_{\omega}) = \dim((A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau}_{\omega}) \;=\; \dim((A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}})\,.$$

The purity of branch locus principle states that if the quotient for a finite group action on a smooth variety is smooth, then the branch loci are all of codimension one. We apply the purity of branch locus principle to $(A \otimes_{\mathbb{Z}} \Lambda_T)/\omega\dot{\tau}$ that is smooth, since the quotient is a group. The branch loci of $(A \otimes_{\mathbb{Z}} \Lambda_T)/\omega\dot{\tau}$ have the same dimension that $(A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\dot{\tau}}$. $\square$

We now proceed to describe the fixed locus $\dot{B}^{\omega\sigma_\epsilon}$. Using the basis $\{\lambda_1, \cdots, \lambda_s\}$ of Λ one gets an isomorphism between $\dot{B}$ and $\overbrace{A \times \cdots \times A}^{s-\text{times}}$. Denote by $M \in \mathrm{GL}(s, \mathbb{Z})$ the matrix of $\omega\sigma_\epsilon$ in this base. We denote by $\bar{t} = (t_1, \cdots, t_s)$ the involution on A^s given by the t_i. We observe that $\omega\dot{\tau}$ corresponds with $M \circ \bar{t}$. It is then clear that

$$(A \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\sigma_\epsilon} \;\cong\; \left(\overbrace{A \times \cdots \times A}^{s-\text{times}}\right)^{M\circ\bar{t}}. \tag{4.5.34}$$

The following is a consequence of Propositions 4.4.1 and 4.5.1, Lemma 4.3.1 and Remarks 4.2.3, 4.2.4 and 4.3.2.

Corollary 4.5.2 Let $t_y \circ \alpha_{(\epsilon,a)}$ be a holomorphic (respectively, anti-holomorphic) involution on X and σ_ϵ a holomorphic (respectively, anti-holomorphic) involution of the complex reductive Lie group G. The fixed locus for the involution $I(t_y \circ \alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \mathcal{F})$ of $\mathcal{M}(G)_{\mathrm{red}}$ is the union

$$\mathcal{M}(G)^{I(t_y\circ\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})}_{\mathrm{red}} \;=\; \bigcup_{\overline{\omega}\in W/_{\sigma_\epsilon}W} \left(T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T\right)^{\omega\left(\dot{I}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})\right)} /N_W(T^{\omega\sigma_\epsilon})\,,$$

where ω is a representative of $\overline{\omega} \in W/_{\sigma_\epsilon}W$. Furthermore, if $M_\omega \in \mathrm{GL}(s, \mathbb{Z})$ is the automorphism $\omega\sigma_\epsilon(\omega)$ expressed in a certain basis of Λ_T, and if $y_1, \cdots, y_s \in T^*\widehat{X}$ are the coordinates of $\mathcal{F} \in T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T$ in this basis, then

$$\left(T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T\right)^{\omega\left(\dot{I}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\mathcal{F})\right)} \;\cong\; \left(\overbrace{T^*\widehat{X} \times \cdots \times T^*\widehat{X}}^{s-\text{times}}\right)^{M_\omega\circ\bar{t}},$$

where

$$\bar{t} \;=\; \big((\nu_{y_1}\circ\alpha_{(+,a)}, \pm\mathrm{Id}), \cdots, (\nu_{y_s}\circ\alpha_{(+,a)}, \pm\mathrm{Id})\big)$$

if $\epsilon = +$, and

$$\bar{t} \;=\; \big((\nu_{y_1}\circ\alpha_{(-,-a)}, \mp\mathrm{conj}), \cdots, (\nu_{y_s}\circ\alpha_{(-,-a)}, \mp\mathrm{conj})\big)$$

if $\epsilon = -$.

Analogously, the fixed locus for the involution $J(t_y\circ\alpha_{(\epsilon,a)}, \sigma_\epsilon, \pm, \chi)$ of $\mathcal{R}(G)_{\mathrm{red}}$ is the union

$$\mathcal{R}(G)_{\mathrm{red}}^{I(t_y\circ\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\chi)} \;=\; \bigcup_{\overline{\omega}\in W/_{\sigma_\epsilon}W} (\mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)\otimes_{\mathbb{Z}}\Lambda_T)^{\omega\left(\dot{J}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\chi)\right)}/N_W(T^{\omega\sigma_+})\,.$$

If the homomorphisms $\chi_i \in \mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)$ are the coordinates of $\chi \in \mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)\otimes_{\mathbb{Z}}\Lambda_T$ in the basis, and if $b_{i,1} := \chi_i(\delta_1)$ and $b_{i,2} := \chi_i(\delta_2)$ are the images of the generators δ_1, δ_2 of $\pi_1(X)$, then

$$(\mathrm{Hom}(\pi_1(X),\, \mathbb{C}^*)\otimes_{\mathbb{Z}}\Lambda_T)^{\omega\left(\dot{J}(\alpha_{(\epsilon,a)},\sigma_\epsilon,\pm,\chi)\right)} \;\cong\; \Big((\mathbb{C}^*\times\mathbb{C}^*)\times \overset{s}{\cdots} \times(\mathbb{C}^*\times\mathbb{C}^*)\Big)^{M_\omega\circ\bar{t}}\,,$$

where

$$\bar{t} \;=\; \left(f_{(\epsilon,a,R)}^{\pm,(b_{1,1},b_{1,2})}, \cdots, f_{(\epsilon,a,R)}^{\pm,(b_{s,1},b_{s,2})}\right)$$

with R being the region of Ω where X lies.

It is straight-forward to check that the subset of W given by the elements of the form $\omega\sigma_\epsilon(\omega)$ is closed under the usual adjoint action of W. We define the quotient

$$\Upsilon_{\sigma_\epsilon} \;=\; \{\omega\sigma_\epsilon(\omega) \;\mid\; \omega\in W\}/\mathrm{ad}(W)\,.$$

Consider the map

$$\delta \,:\, W/_{\sigma_\epsilon}W \;\longrightarrow\; \Upsilon_{\sigma_\epsilon}\,, \quad [\omega]_{\mathrm{ad}_{\sigma_\epsilon}} \;\longmapsto\; [\omega\sigma_\epsilon(\omega)]_{\mathrm{ad}}\,.$$

Fix $\gamma \in \Upsilon_{\sigma_\epsilon}$. We define

$$(A\otimes_{\mathbb{Z}}\Lambda_T/W)^{\tau,\gamma} \;:=\; \bigcup_{\delta([\omega])=\gamma} (A\otimes_{\mathbb{Z}}\Lambda_T/W)^{\tau}_{\omega}\,;$$

it is clear that

$$(A\otimes_{\mathbb{Z}}\Lambda_T/W)^{\tau} \;=\; \bigcup_{\gamma\in\Upsilon_{\sigma_\epsilon}} (A\otimes_{\mathbb{Z}}\Lambda_T/W)^{\tau,\gamma}\,.$$

For a fixed representative of each $\gamma \in \Upsilon_{\sigma_\epsilon}$ one define the non abelian shifted group cohomology $H^1_\gamma(\sigma_\epsilon, W)$ as follows. The γ-shifted cocycle condition is given by

$$\gamma = \omega\sigma_\epsilon(\omega)\,. \tag{4.5.35}$$

The coboundary condition that we take is given by Equation (4.5.30). One has that

$$(A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau,\gamma} = \bigcup_{\overline{\omega}\in H^1_\gamma(\sigma_\epsilon,W)} (A \otimes_{\mathbb{Z}} \Lambda_T/W)^{\tau}_{\overline{\omega}}\,. \tag{4.5.36}$$

4.6 Moduli spaces of pseudo-real Higgs bundles

Every element $c \in Z_2$ of order 2 of the center defines an element of the Weyl group ω_c in the following way (see for instance [33, 35]). Take an alcove $A \subset \mathfrak{t}$ containing the origin. We know (see for instance [26]) that there is a vertex a_c of the alcove A such that $z = \exp(a_c)$. We see that $A - a_c$ is another alcove containing the origin. Hence there is a unique element $\omega_z \in W$ such that

$$A - a_c = \omega_c(A)\,.$$

In the trivial case we obviously have $\omega_0 = \mathrm{Id}$.

Remark 4.6.1 Note that the action of ω_c on T coincides with the action of z on it.

We denote by $\mathcal{M}(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm)$ the moduli space of $(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm)$-real G-Higgs bundles. (See Example 2.1.1). We denote by $\mathcal{R}(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm)$ the moduli space of $(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm)-$compatible representations of the orbifold fundamental group of X. (See Definition 1.4.3). Now, we can provide a description of the moduli spaces of pseudo-real G-Higgs bundles. We denote by $\widetilde{\mathcal{M}}(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm)_{\mathrm{red}}$ the intersection of $\mathcal{M}(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm,)$ and $\mathcal{M}(G)_{\mathrm{red}}$. We denote by $\widetilde{\mathcal{R}}(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm)_{\mathrm{red}}$ the intersection of $\mathcal{R}(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm)$ and $\mathcal{R}(G)_{\mathrm{red}}$.

Theorem 4.6.1 Take the central element $c \in Z_2^{\sigma_-}$, the antiholomorphic involution

$$t_y \circ \alpha_{(-,a)} : X \longrightarrow X\,,$$

and the pair of holomorphic and anti-holomorphic involutions $\sigma_+, \sigma_- : G \longrightarrow G$ such that $\sigma_+ = \sigma_-\tau$. The image of the reduced moduli space of $(\alpha_{(-,a)}, \sigma_-, c, \pm)$-Higgs bundles under the forgetful morphism is

$$\widetilde{\mathcal{M}}(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm)_{\mathrm{red}} = \bigcup_{\overline{\omega} \in H^1_{\omega_c}(\sigma_-, W)} \left(T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T\right)^{\omega\left(\dot{I}(\alpha_{(-,a)}, \sigma_-, \pm)\right)} / N_W(T^{\omega\sigma_-}) \,.$$

Analogously, the image of the reduced moduli space of representations is

$$\widetilde{\mathcal{R}}(t_y \circ \alpha_{(-,a)}, \sigma_-, c, \pm)_{\mathrm{red}} = \bigcup_{\overline{\omega} \in H^1_{\omega_c}(\sigma_-, W)} (\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \otimes_{\mathbb{Z}} \Lambda_T)^{\omega\left(\dot{J}(\alpha_{(-,a)}, \sigma_-, \pm)\right)} / N_W(T^{\omega\sigma_-}) \,.$$

Proof: In view of Table 1.2, without any loss of generality, we can take $\alpha_{(\epsilon,a)}$ to be our anti-holomorphic involution. From Theorem 3.6.1 we have that $\widetilde{\mathcal{M}}(\alpha_{(-,a)}, \sigma_-, c, \pm)_{\mathrm{red}}$ lies in the fixed-point locus of $I(\alpha_{(-,a)}, \sigma_-, \pm)$. In fact, recalling Remark 4.6.1, $\widetilde{\mathcal{M}}(G, \alpha_{(-,a)}, \sigma_-, \pm, z)_{\mathrm{red}}$ are those lying in the components $(T^*\widehat{X} \otimes_{\mathbb{Z}} \Lambda_T / W)_{\omega}^{\dot{I}(\alpha_{(-,a)}, \sigma_-, \pm)}$ where

$$\omega\sigma_-(\omega) = \omega_c.$$

From this and (4.5.36), we see that $\widetilde{\mathcal{M}}(\alpha_{(-,a)}, \sigma_-, c, \pm)$ is given by the union of the components of $H^1_{\omega_c}(\sigma_-, W)$, and the first statement follows from Corollary 4.5.2.

Finally, the second statement follow from the previous description and [14, Theorems 4.5 and 4.8]. □

Remark 4.6.2 Recall that for $c = \mathrm{Id}_G$, one has $\omega_c = \mathrm{Id}_W$, and therefore the moduli space of real $(G, \alpha_-, \sigma_-, \pm)$-Higgs bundles (corresponding to $c = \mathrm{Id}_G$) is the union of the components classified by $H^1(\sigma_-, W) = H^1_{\mathrm{Id}}(\sigma_-, W)$, the non-abelian group cohomology associated to the action of $\mathbb{Z}/2\mathbb{Z}$ on W given by σ_ϵ.

Chapter 5

Real parabolic G-Higgs bundles

Along this chapter, we use the following notation:

- $X-$compact Riemann surface
- $S-$a finite set of different points in X
- $G-$real form of a semisimple Lie group $G^{\mathbb{C}}$
- $\mu-$conjugation of $G^{\mathbb{C}}$ such that $(G^{\mathbb{C}})^{\mu} = G$
- $\tau-$conjugation of $G^{\mathbb{C}}$ such that $(G^{\mathbb{C}})^{\tau} = H^{\mathbb{C}}$ is a maximal compact subgroup of $G^{\mathbb{C}}$
- $\theta := \mu \circ \tau$
- σ_G-conjugation of $G^{\mathbb{C}}$ that is $(\mu, \tau)-$compatible
- $Z-$center of $H^{\mathbb{C}}$
- $Z_2^{\sigma_G}-$subgroup of elements of Z of order two that are invariants under σ_G
- $\mathfrak{g}, \mathfrak{h}, \mathfrak{z}-$Lie algebras of G, H, Z, respectively
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the Cartan decomposition
- $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$ the isotropy representation
- $c \in Z_2(H^{\mathbb{C}})^{\sigma_G} \cap Z(G^{\mathbb{C}})$.

5.1 Parabolic G-Higgs bundles

In this section, we recall some notions of the theory of parabolic G-Higgs bundles, following [22].

5.1.1 Moduli space of parabolic G-Higgs bundles

Let G be a real form of a reductive Lie group $G^{\mathbb{C}}$ and μ be a conjugation of $G^{\mathbb{C}}$ such that the fixed point set $(G^{\mathbb{C}})^{\mu}$ of $G^{\mathbb{C}}$ under μ is equal to G. Let τ be a compact conjugation of $G^{\mathbb{C}}$, i.e. $H^{\mathbb{C}} := (G^{\mathbb{C}})^{\tau}$ is a maximal compact subgroup of $G^{\mathbb{C}}$. Let σ_G be a (μ, τ)-compatible conjugation of $G^{\mathbb{C}}$. (Definition 1.3.10). Let T be a maximal torus of H . We denote by $\mathfrak{t}$ its Lie algebra. Let $W := N(T)/T$ be the Weyl group of H with respect to T. The **co-character lattice** $\Lambda_{\text{cochar}} \subset \mathfrak{t}$ is the kernel of the exponential map $\exp : \mathfrak{t} \to T$. The **affine Weyl group** is the semidirect product

$$W_{\text{aff}} = \Lambda_{\text{cochar}} \rtimes W\,,$$

where Λ_{cochar} acts on $\mathfrak{t}$ by translations. An **alcove** is a point in $\mathfrak{t}/W_{\text{aff}}$. This quotient space is homeomorphic to the following spaces:

$$\mathfrak{t}/W_{\text{aff}} \simeq T/W \simeq \text{Conj}(H)\,, \tag{5.1.1}$$

where $\text{Conj}(H)$ denotes the equivalence classes on H given by the relation $\sim$ that is defined as follows: $h_1 \sim h_2$ if and only if there is $h \in H$ such that $h_1 = h_1^{-1}\, h\, h_2$.

We fix an alcove $\mathcal{A} \subset \mathfrak{t}$ containing $0 \in \mathfrak{t}$. Let $\bar{\mathcal{A}}$ be its clousure on $\mathfrak{t}$. Let X be a compact connected Riemann surface and $S = \{x_1, \dots, x_r\}$ be a finite set of different points of X. Let D be the divisor $x_1 + \cdots + x_n$. Let E be a principal $H^{\mathbb{C}}$-bundle.

Definition 5.1.1 A **weight** on E over $x_i \in S$ is an element $\alpha_i \in \sqrt{-1}\,\bar{\mathcal{A}}$.

Definition 5.1.2 A **parabolic structure of weight** α_i **on** E **over a point** x_i is a choice of a subgroup

$$Q_i \subset E(H^{\mathbb{C}})_{x_i} = \{\, \phi : E_{x_i} \to H^{\mathbb{C}}, \phi(eg) = g^{-1}\phi(e)g \;\; \forall e \in E_x,\, g \in H^{\mathbb{C}} \}\,, \tag{5.1.2}$$

such that in some trivialization $e \in E_{x_i}$, we have

$$P_{\alpha_i} = \{\phi(e),\;\; \phi \in Q_i\}\,, \tag{5.1.3}$$

where P_{α_i} is the parabolic subgroup of $H^{\mathbb{C}}$ defined in Section [2.3].

Definition 5.1.3 Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a r-tuple of weights. A **parabolic principal bundle of weight** α is a holomorhic $H^{\mathbb{C}}$-bundle E equipped with a parabolic structure of weight α_i on x_i, for every $x_i \in S$.

Definition 5.1.4 The **sheaf of groups** $\mathcal{G}_\alpha$ on X is defined as follows: for every open subset $\Omega \subset X$, $\mathcal{G}_\alpha(\Omega)$ is the set of all holomorphic maps $\phi : \Omega^* \to H^{\mathbb{C}}$ satisfying that for every j with $x_j \in \Omega$,

$$|z_j(p)|^{-\alpha_j}\phi(p)|z_j(p)|^{\alpha_j}$$

stays in a compact subset of $H^{\mathbb{C}}$ as $p \to x_j$.

Definition 5.1.5 Two parabolic bundles $(E_0, \{Q_{0,i}\})$ and $(E_1, \{Q_{1,i}\})$ are **meromorphically equivalent** if there is an isomorphism of principal bundles

$$\psi : E_0|_{X^*} \to E_1|_{X^*}\,,$$

such that $f(\psi) \in \mathcal{G}_\alpha$, where $f(\psi) : U \setminus \{x_j\} \to H^{\mathbb{C}}$ is defined as follows. Let $U \subset X$ be a small disk centred at x_j and disjoint from all other points of the support of D. Let $\sigma_i \in \Gamma(U, E_i)$, $i = 0, 1$, be holomorphic sections satisfying

$$\sigma_i(x_j) \in R_{i,j} := \{e \in E_i,\ \ P_{\alpha_i} = \{\phi(e), \phi \in Q_j\}\}\,.$$

Then $f(\psi)$ is defined by

$$\psi(\sigma(y)) = \sigma'(y) f(\psi)(y) \ \text{ for every } y \in U \setminus \{x_j\}\,.$$

Let $P^1_{\alpha_i}$ be the unipotent subgroup of P_{α_i} such that its lie algebra is defined by

$$\mathfrak{p}^1_{\alpha_i} = \ker(\mathrm{ad}(\alpha_i) + 1)\,.$$

Let Q^1_i be the normal abelian subgroup of Q_i given by

$$Q^1_i = \{\phi \in E(H^{\mathbb{C}})_{x_i} \mid \phi(e) \in P^1_{\alpha_i}\}\,.$$

Let $\mathfrak{q}^1_i$ be the Lie algebra of Q^1_i.

Definition 5.1.6 The **parabolic group of gauge transformations** $PE(H^{\mathbb{C}})$ **of a parabolic bundle** E is the sheaf whose sections on $X \setminus S$ are the holomorphic sections g of $E(H^{\mathbb{C}})$, and near a marked point x_i, the sections are of the form $g(z)\exp(n/z)$ in some trivialization near x_i, such that $n \in \mathfrak{q}_i^1$ and g is holomorphic near x_i with $g(0) \in Q_i$.

Let $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$ be the isotropy representation defined in 3.1.1 and $E(\mathfrak{m}^{\mathbb{C}})$ be the associated bundle. Let $\alpha_i \in \bar{\mathcal{A}}$ be the parabolic weight at x_i. In the trivialization e_i, we can decompose the bundle $E(\mathfrak{m}^{\mathbb{C}})$ under the eigenvalues of $\mathrm{ad}(\alpha_i)$ (acting on $\mathfrak{m}^{\mathbb{C}}$),

$$E(\mathfrak{m}^{\mathbb{C}}) = \oplus_\mu \mathfrak{m}^{\mathbb{C}}_\mu \,. \tag{5.1.4}$$

Definition 5.1.7 The **sheaf** $PE(\mathfrak{m}^{\mathbb{C}})$ **of parabolic sections of** $E(\mathfrak{m}^{\mathbb{C}})$ (resp. **sheaf** $NE(\mathfrak{m}^{\mathbb{C}})$ **of strictly parabolic sections of** $E(\mathfrak{m}^{\mathbb{C}})$) are holomorphic sections φ of $E(\mathfrak{m}^{\mathbb{C}})$ on $X \setminus S$ such that if $\varphi = \sum \varphi_\mu$, according to the descomposition 5.1.4, then φ_μ is meromorphic on x_i and $\varphi_\mu = O(z^a)$, with $a-1 < \mu \leq a$, (resp. $a-1 \leq \mu < a$) for some integer a.

Definition 5.1.8 A **parabolic (resp. strictly parabolic) Higgs bundle** (E,φ) **on** (X,D) is a parabolic principal bundle such that $\varphi \in PE(\mathfrak{g}) \otimes K(D)$ (resp. $\varphi \in NE(\mathfrak{g}) \otimes K(D)$).

Let $s \in \sqrt{-1}\mathfrak{h}$ and χ_s be the corresponding antidominant character of $\mathfrak{p}_s$. Consider the decomposition $\mathfrak{h} = \mathfrak{z} + [\mathfrak{h},\mathfrak{h}]$ of the Lie algebra of H. Let $\mathfrak{z}' = \ker(d\iota_{|\mathfrak{z}})$ and take $\mathfrak{z}''$ such that $\mathfrak{z} = \mathfrak{z}' + \mathfrak{z}''$. We define $\mathfrak{h}_\iota := \mathfrak{z}'' + [\mathfrak{h},\mathfrak{h}]$. Finally, we consider

$$\mathfrak{m}_s := \{v \in \mathfrak{m}^{\mathbb{C}} \mid \mathrm{Ad}(e^{ts})v \text{ is bounded as } t \to \infty\}.$$
$$\mathfrak{m}^0_s := \{v \in \mathfrak{m}^{\mathbb{C}} \mid \mathrm{Ad}(e^{ts})v = v \text{ for every } t\}.$$

We give two equivalent definitions of pardeg in the Appendix.

Definition 5.1.9 Let $z \in \sqrt{-1}\mathfrak{z}$. A parabolic G-Higgs bundle (E,φ) is:

- z-**semistable** if for every reduction of E to P_s such that $\varphi|_{X\setminus D} \in H^0(X\setminus S, E_{P_s}(\mathfrak{m}_s)\otimes K)$, then

$$\mathrm{pardeg}_\alpha(E)(s,\sigma) - \langle z, s\rangle \geq 0\,,$$

- z-**stable** if we restrict $s \in \sqrt{-1}\mathfrak{h}_i$, then for every reduction of E to P_s such that $\varphi|_{X\backslash D} \in H^0(X\backslash S, E_{P_s}(\mathfrak{m}_s) \otimes K)$, we have

$$\operatorname{pardeg}_\alpha(E)(s,\sigma) - \langle z, s\rangle \;>\; 0\,,$$

- z-**polystable** if it is z-semistable and if equality occurs for some $s \in \sqrt{-1}\mathfrak{h}_i$ and σ then there is $g \in PE(H^{\mathbb{C}})$ such that $g(E,\varphi) = (E',\varphi)$ admits a holomorphic reduction σ_{L_s} to L_s such that:

  (1) $\varphi|_{X\backslash D} \in H^0(X\backslash D, E_{L_s}(\mathfrak{m}_s^0) \otimes K)$

  (2) E_{L_s} has a parabolic structure such that E' is induced from E_{L_s} through the injection $L_s \subset H^{\mathbb{C}}$.

We denote by $\mathcal{M}_z(\alpha)$ the moduli space of meromorphic equivalence classes of z-polystable parabolic G-Higgs bundles.

Definition 5.1.10 $\operatorname{Gr}\operatorname{Res}_{x_i}\varphi$ is the projection of the residue of φ at x_i to the space $\widetilde{\mathfrak{m}_i^0} \subset E(\mathfrak{m}^{\mathbb{C}})_{x_i}$ that corresponds to

$$\widetilde{\mathfrak{m}_{\alpha_i}^0} := \operatorname{Ker}_{\mathfrak{m}^{\mathbb{C}}}(\operatorname{Ad}(\exp 2\pi\sqrt{-1}\alpha_i) - 1)\,, \tag{5.1.5}$$

via the isomorphism induced by the trivialization e_i that identifies Q_i with P_{α_i}, after choosing a trivialization of $\mathcal{O}(D)$ at x_i.

Let $\widetilde{L}_i$ be the corresponding subspace of Q_i to

$$\widetilde{L_{\alpha_i}} := \operatorname{Stab}_{H^{\mathbb{C}}}(e^{2\pi\sqrt{-1}\alpha_i}) \tag{5.1.6}$$

under the isomorphism of Q_i with P_{α_i} given by the parabolic structure of E. Consider the map

$$\begin{array}{rccc} \rho: & \mathcal{M}_c(\alpha) & \longrightarrow & \prod_i(\widetilde{\mathfrak{m}_i^0}/\widetilde{L}_i) \\ & (E,\varphi) & \longmapsto & [\operatorname{Gr}\operatorname{Res}_{x_i}\varphi]\,. \end{array} \tag{5.1.7}$$

Definition 5.1.11 Fix classes $\mathcal{L}_i \in \widetilde{\mathfrak{m}_i^0}/\widetilde{L}_i$ with $1 \le i \le r$ and let $\mathcal{L} := (\mathcal{L}_1, \dots, \mathcal{L}_r)$. The moduli space $\mathcal{M}_z(\alpha,\mathcal{L}) := \rho^{-1}(\mathcal{L})$.

Definition 5.1.12 A parabolic G-Higgs bundle (E,φ) is **simple** if $\operatorname{Aut}(E,\varphi) = Z(H^{\mathbb{C}}) \cap \operatorname{Ker}(\iota)$.

We denote by $\mathcal{M}_z(\alpha, \mathcal{L})_{\mathrm{sm}}$ the moduli space of simple, z-stable and with vanishing obstruction class.

5.1.2 Moduli space of parabolic G-local systems

Definition 5.1.13 Let $\beta = (\beta_1, \ldots, \beta_r)$ be a r-tuple of elements $\beta_i \in \mathfrak{m}$, for $1 \leq i \leq r$. A **parabolic G-local system of weight** $\beta = (\beta_1, \ldots, \beta_r)$ is a G-bundle equipped with a flat connection over $X \backslash S$, such that for every x_i and every ray ρ_i going to x_i, there is a pair (P_i, χ_i), where P_i is a parabolic subgroup of $F(G)|_{\rho_i}$ isomorphic via trivialization 5.1.3 to P_{β_i}, invariant under the monodromy transformation around x_i and χ_i is a strictly antidominant character of P_i.

Consider the fixed points of the isotropy action on $\mathfrak{m}$

$$\mathfrak{m}^H := \{v \in \mathfrak{m} \,|\, \mathrm{Ad}(h)(v) = v \quad \text{for every } h \in H\} .$$

Definition 5.1.14 Let $\zeta \in \mathfrak{m}^H$. Let $Q \subset G$ be a parabolic subgroup of G and χ be an antidominant character of its Lie algebra $\mathfrak{q}$. We say that a parabolic G-local system of weight $\beta = (\beta_1, \ldots, \beta_r)$ is:

- **ζ-semistable** if for any reduction F_Q of the local system invariant under the flat connection, one has

$$\mathrm{pardeg}(F)(Q, \chi, \sigma) := -\sum_i \deg\big((P_i, s_{\chi_i}), (Q, s_\chi)\big) - \langle \zeta, s_\chi \rangle \geq 0 \,, \qquad (5.1.8)$$

  where s_χ, is the dual of χ under an H-invariant inner product $\langle , \rangle$ on $\mathfrak{h}$, recall the definition of relative degree in Equation (A.2.3) of the Appendix.

- **ζ-stable** if the inequality is strict for any non-trivial reduction,

- **ζ-polystable** if it is semistable and equality happens in (5.6.26) if and only if there is a reduction of the local system to a Levi subgroup $L \subset Q$, meaning that there is a parabolic L-local system which induces F via the inclusion $L \subset G$.

We denote by $\mathcal{S}_\zeta(\beta)$ the moduli space of isomorphism classes of ζ-polystable parabolic G-local systems of weights β.

The monodromy of the loop c_i around x_i takes values in P_{β_i}, and we consider its projection to L_{β_i}. Its conjugacy class $\mathcal{C}_i$ in L_{β_i} is independent of the simple loop that we have taken. This defines a map

$$\mu : \mathcal{S}_\zeta(\beta) \longrightarrow \prod_i \mathrm{Conj}(L_{\beta_i}), \tag{5.1.9}$$

where $\mathrm{Conj}(L_{\beta_i})$ is the set of conjugacy classes of L_{β_i}. Let $\mathcal{C} = (\mathcal{C}_1, \cdots, \mathcal{C}_r)$ with $\mathcal{C}_i \in \mathrm{Conj}(L_{\beta_i})$, and consider the moduli space

$$\mathcal{S}_\zeta(\beta, \mathcal{C}) = \mu^{-1}(\mathcal{C}).$$

Definition 5.1.15 A parabolic G-local system is **irreducible** if its group of automorphisms is equal to $Z(G)$.

We denote by $\mathcal{S}_\zeta(\beta, \mathcal{C})_{\mathrm{sm}}$ the moduli space of stable and irreducible elements.

5.1.3 Moduli space of representations of $\pi_1(X\backslash D)$

Definition 5.1.16 Let $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_r\}$ an r-tuple of conjugacy classes of G. A **$\mathcal{C}$-representation of $\pi_1(X \setminus D)$ in G** is a homomorphism $\rho \colon \pi_1(X \setminus S) \to G$ such that the image of any class of a loop around x_i is $\mathcal{C}_i$, for $1 \leq i \leq r$.

Definition 5.1.17 A $\mathcal{C}$-representation is **reductive** if composed with the adjoint representation in $\mathfrak{g}$ decomposes as a sum of irreducible representations.

The **moduli space of reductive $\mathcal{C}$-representations of $\pi_1(X \setminus D)$ in G** is

$$\mathcal{R}(\mathcal{C}) := \mathrm{Hom}^+(\pi_1(X \setminus D), G)/G,$$

where G acts by conjugation. We denote by $\mathcal{R}(\mathcal{C})_{\mathrm{sm}}$ the moduli space of irreducible representations.

5.1.4 Relations between moduli spaces

The following Table 5.1 relates the weights and the monodromies of a Higgs parabolic bundle and a parabolic local system.

	Weight	**Monodromy (projected to the Levi)**
(E,φ)	α	$s+Y=\mathrm{Gr}\,\mathrm{Res}_x\,\varphi$
(F,∇)	$\beta=s-\tau(s)$	$\exp(2\pi\sqrt{-1}\alpha)\exp(2\pi\sqrt{-1}(-s-\tau(s)+Y-H-X))$

Table 5.1: Weights and monodromies

Theorem 5.1.18 ([22], Propositions 7.2 and 7.7) If the weights and the monodromies are related by Table 5.1, then:

(1) $\mathcal{M}(\alpha,\mathcal{L})$ is homeomorphic to $\mathcal{S}(\beta,\mathcal{C})$.

(2) $\mathcal{M}^*(\alpha,\mathcal{L})$ is differomorphic to $\mathcal{S}^*(\beta,\mathcal{C})$.

(3) $\mathcal{S}(\beta,0)$ is homeomorphic to $\mathcal{R}(\beta)$.

(4) $\mathcal{S}^*(\beta,0)$ is diffeomorphic to $\mathcal{R}(\beta)$.

5.2 Involutions of the moduli spaces

Proposition 5.2.1 There is a maximal torus $T\subset H$ invariant under σ_G, furthermore there is an alcove $\mathcal{A}\subset\mathfrak{t}$ invariant under $d\sigma_G$, such that $0\in\overline{\mathcal{A}}$.

Proof: By [23] Theorem 1 p. 329, in every connected and compact Lie group H there is a maximal torus $T\subset H$ and a chamber associated $C(T)$, such that they are both invariant under σ_G. The maximal compact subgroup H is connected because G is connected. On the other hand, by [23] Proposition 2, d) p. 326 any chamber contains a unique alcove whose closure contains 0. Then, let $\mathcal{A}$ be the unique alcove whose closure contains 0 contained in $C(T)$. The uniqueness forces to be invariant under $d\sigma_G$. □

Let (X,σ_X) be a compact Klein surface and E be a holomorhic $H^{\mathbb{C}}$-bundle. We consider the holomorhic $H^{\mathbb{C}}$-bundle $E':=\sigma_X^*E\times_{\sigma_G}H^{\mathbb{C}}$. Let

$$f:E\to E'$$

be the lift of σ_X defined in 3.5.10. The conjugation σ_G and f induce a map between the adjoint bundles

$$\mathrm{Ad}(f):E(H^{\mathbb{C}})\to E'(H^{\mathbb{C}})\,. \tag{5.2.10}$$

Fix an invariant alcove $\mathcal{A}$, (see Proposition 5.2.1). Let $x \in X$ be a prescribed point and α be a a weight over x. By Equation (5.1.2) in the definition of parabolic structures, and by an analogous result to Proposition 1.7.1, one has

$$\mathrm{Ad}(f)(\phi) = \sigma_G \circ \phi \circ f^{-1}\,, \tag{5.2.11}$$

for all $\phi \in E(H^{\mathbb{C}})_x$.

Proposition 5.2.2 The map $\mathrm{Ad}(f)$ sends parabolic structures $Q \subset E(H^{\mathbb{C}})_x$ of weight α on E over x to parabolic structures $Q' := \mathrm{Ad}(f)(Q) \subset E'(H^{\mathbb{C}})_{\sigma_X(x)}$ of weight $\sigma_G(\alpha)$ on E' over $\sigma_X(x)$.

Proof: Let $Q \subset E(H^{\mathbb{C}})$ be a parabolic structure of weight α on E over x. By Definition 5.1.2 there is a trivialization $e \in E_x$ such that

$$P_\alpha = \{\phi(e),\ \ \phi \in Q\}\,. \tag{5.2.12}$$

Since $P_{\sigma_G(\alpha)} = \sigma_G(P_\alpha)$, then $P_{\sigma_G(\alpha)} = \{\sigma_G \circ \phi(e),\ \ \phi \in Q\}$. Now, choosing the trivialization $e' = f(e) \in E_{\sigma_X(x)}$, we obtain

$$P_{\sigma_G(\alpha)} = \{\sigma_G \circ \phi \circ f^{-1}(e'),\ \ \phi \in Q\} = \{\phi'(e'),\ \ \phi' \in \mathrm{Ad}(f)(Q)\}\,.$$

Therefore, $Q' = \mathrm{Ad}(f)(Q)$ is a parabolic structure of weight $\sigma_G(\alpha)$ on E' over $\sigma_X(x)$. $\square$

From Proposition A.1.1, it follows that σ_G is an isometry with respect to the Tits distance, so f preserves the relative degree and pardeg. It also sends reductions of the structure group of E from $H^{\mathbb{C}}$ to P_s in reductions of the structure group of E' from $H^{\mathbb{C}}$ to $P_{\sigma_G(s)}$. So f preserves polystability. Since σ_G commutes with θ, the map $\mathrm{Ad}(f)$ defined in (5.2.11), restricts to

$$\mathrm{Ad}(f) : E(\mathfrak{m}^{\mathbb{C}}) \to E'(\mathfrak{m}^{\mathbb{C}})\,. \tag{5.2.13}$$

In addition, it preserves the order of the poles, so $\mathrm{Ad}f$ and $\sigma_X : K(D) \to K(\sigma_X D)$ induce a map

$$P\mathrm{Ad}(f) : PE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D) \to PE'(\mathfrak{m}^{\mathbb{C}}) \otimes K(\sigma_X(D))\,, \tag{5.2.14}$$

that sends a Higgs field $\varphi \in PE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)$ in a Higgs field $\varphi' := P\mathrm{Ad}(f)(\varphi) \in PE'(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)$. We can see that f also preserves the meromorphic equivalence of parabolic bundles defined in (5.1.5). As a result, from Proposition 5.2.2, it follows that f induce maps

$$\begin{array}{rccl} f^{\pm}: & \mathcal{M}_z(\alpha) & \longrightarrow & \mathcal{M}_z(\sigma_G(\alpha)) \\ & (E,\varphi) & \longmapsto & (E',\, \pm\varphi'\,)\,. \end{array} \tag{5.2.15}$$

Taking fibers in (5.2.13), we obtain

$$\mathrm{Ad}(f^{\pm})(\widetilde{\mathfrak{m}^0_\alpha}) = \sigma_G(\widetilde{\mathfrak{m}^0_\alpha}) = \widetilde{\mathfrak{m}^0_{\sigma_G(\alpha)}}\,,$$

and also $\mathrm{Ad}(f^{\pm})(\widetilde{L_\alpha}) = \sigma_G(\widetilde{L_\alpha}) = \widetilde{L_{\sigma_G(\alpha)}}$. Then, there is a map

$$\mathrm{Ad}(f^{\pm}) : \widetilde{\mathfrak{m}^0_\alpha}/\widetilde{L_\alpha} \to \widetilde{\mathfrak{m}^0_{\sigma_G(\alpha)}}/\widetilde{L_{\sigma_G(\alpha)}}\,.$$

If z is a local coordinate near x, then in a holomorphic trivialization e of E near x_i, we can write

$$\varphi = \sum_{\mu\in\mathbb{Z}} \frac{\varphi_\mu}{z^{\lfloor -\mu \rfloor}} \frac{dz}{z}\,,$$

where φ_μ is a holomorphic function, and then

$$\mathrm{Gr}\,\mathrm{Res}_{x_i}\,\varphi = \sum_{\mu\in\mathbb{Z}} \varphi_\mu(0)\,.$$

If we take the coordinates, $z' = f^{\pm}(z)$, then near $\sigma_X(x)$,

$$P\mathrm{Ad}(f^{\pm})(\varphi) = \sum_{\mu\in\mathbb{Z}} \frac{\mathrm{Ad}(f^{\pm})(\varphi_\mu)}{z'^{\lfloor -\mu \rfloor}} \frac{dz'}{z'}\,,$$

where $P\mathrm{Ad}(f^{\pm})$ is defined in (5.2.14) and

$$\mathrm{Gr}\,\mathrm{Res}_{\sigma_X(x)}\, P\mathrm{Ad}(f^{\pm})(\varphi) = \sum_{\mu\in\mathbb{Z}} \mathrm{Ad}(f^{\pm})(\varphi_\mu)(0) = \mathrm{Ad}(f^{\pm})(\mathrm{Gr}\,\mathrm{Res}_x\,\varphi)\,. \tag{5.2.16}$$

Consider the map

$$\begin{array}{rccl} \rho: & \mathcal{M}_z(\sigma_X,\sigma_G,\alpha,\pm) & \longrightarrow & \prod_i(\widetilde{\mathfrak{m}^0_i}/\widetilde{L_i}) \\ & (E,\sigma_E,\varphi) & \longmapsto & [\mathrm{Gr}\,\mathrm{Res}_{x_i}\,\varphi]\,. \end{array} \tag{5.2.17}$$

By (5.2.16), the following diagram is commutative

$$\begin{tikzcd} \mathcal{M}_z(\alpha) \arrow[r, "\rho"] \arrow[d, "f^{\pm}"'] & \widetilde{\mathfrak{m}^0_\alpha}/\widetilde{L_\alpha} \arrow[d, "\mathrm{Ad}(f^{\pm})"] \\ \mathcal{M}_z(\sigma_G(\alpha)) \arrow[r, "\rho"] & \widetilde{\mathfrak{m}^0_{\sigma_G(\alpha)}}/\widetilde{L_{\sigma_G(\alpha)}}\,. \end{tikzcd} \tag{5.2.18}$$

Definition 5.2.1 A set of points $S = \{x_1, \ldots, x_r\} \subset X$ is **real** if $\sigma_X S = S$.

Definition 5.2.2 Let S be a real set of r points of X, $A = \prod_{i=1}^r A_i$ be a product of sets and let

$$\sigma_A : \prod_{i=1}^r A_i \to \prod_{i=1}^r A_{\sigma(i)}$$

be a map, where $\sigma(i)$ is the natural number satisfying $x_{\sigma(i)} = \sigma_X(x_i)$, for all $1 \leq i \leq r$ An element $(a_1, \ldots, a_r) \in A$ is (S, σ_A)-**compatible** if

$$\sigma_A(a_1, \ldots, a_r) = (a_{\sigma(1)}, \ldots, a_{\sigma(r)})\,.$$

Proposition 5.2.3 Let S be a real set of r points of X and $\alpha = (\alpha_1, \ldots, \alpha_r) \in \prod_{i=1}^r \sqrt{-1}\bar{\mathcal{A}}$ be a (S, σ_G)-real element, where $\mathcal{A}$ is an alcove invariant under σ_G. (See Proposition 5.2.1). In this case, the map $f^{\pm} : \mathcal{M}_z(\alpha) \to \mathcal{M}_z(\alpha)$ defined on (5.2.15) is an involution of moduli spaces. Let $\mathcal{L} := (\mathcal{L}_1, \ldots, \mathcal{L}_r) \in \prod_{i=1}^r \widetilde{\mathfrak{m}_i^0}/\widetilde{L}_i$ be $(S, \mathrm{Ad}(f^{\pm}))$-real element, then the map $f^{\pm}$ restricts to an involution

$$i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X, \sigma_G)^{\pm} : \mathcal{M}_z(\alpha, \mathcal{L}) \to \mathcal{M}_z(\alpha, \mathcal{L})\,.$$

Proof: It is a consequence of the commutativity of the diagram (5.2.18). By hypothesis $\mathcal{L}$ is a fixed point of $\mathrm{Ad}(f)$, then its antiimage is fixed by $f^{\pm}$. $\square$

Let S be a real set of r points of X and $\beta = (\beta_1, \ldots, \beta_r) \in \prod_{i=1}^r \mathfrak{t}$ be a (S, σ_G)-real element. We define on $\mathcal{S}(\beta)$ the involutions

$$\begin{array}{rccc} i_{\mathcal{S}(\beta)}(\sigma_X, \sigma_G)^{\pm} : & \mathcal{S}(\beta) & \longrightarrow & \mathcal{S}(\beta) \\ & D & \longmapsto & (\sigma_X)^* \sigma_G \circ \tau^{-\frac{1}{2} \pm \frac{1}{2}}(D)\,. \end{array} \tag{5.2.19}$$

Proposition 5.2.4 If the weights and the monodromies are related by Table 5.1, then the following diagram commutes

$$\begin{tikzcd} \mathcal{M}(\alpha, \mathcal{L}) \arrow[r, "\text{homeo.}"] \arrow[d, "i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X,\sigma_G)^{\pm}"'] & \mathcal{S}(\beta, \mathcal{C}) \arrow[d, "i_{\mathcal{S}(\beta,\mathcal{C})}(\sigma_X,\sigma_G)^{\pm}"] \\ \mathcal{M}(\alpha, \mathcal{L}) \arrow[r, "\text{homeo.}"] & \mathcal{S}(\beta, \mathcal{C})\,. \end{tikzcd}$$

Proof: Outside S, Proposition 5.2.4 is proved in Proposition 3.5.1. A parabolic structure Q_i of weight α_i of E over x_i corresponds to a pair (P_i, χ_i). From Proposition

5.2.2, one has that $i_{\mathcal{M}(\alpha,\mathcal{L})}(\sigma_X,\sigma_G)^{\pm}(Q_i) = Q_{\sigma(i)}$ and since $i_{\mathcal{S}(\beta,\mathcal{C})}(\sigma_X,\sigma_G)^{\pm}(P_i,\chi_i) = (P_{\sigma(i)},\chi_{\sigma(i)})$, because $P_{\beta_{\sigma(i)}} = \sigma_G(P_{\beta_i})$, then the diagram commutes. $\square$

Let S be a real set of r points of X and $\mathcal{C} = (\mathcal{C}_1,\ldots,\mathcal{C}_r) \in \prod_{i=1}^r \mathrm{Conj}(G)$ be a (S,σ_G)-real element. We define on $\mathcal{R}(\mathcal{C})$ the involutions

$$\begin{array}{rccc} i_{\mathcal{R}(G)}(\sigma_X,\sigma_G)^{\pm}: & \mathcal{R}(\mathcal{C}) & \longrightarrow & \mathcal{R}(\mathcal{C}) \\ & \rho & \longmapsto & \sigma_G \circ \tau^{-\frac{1}{2}\pm\frac{1}{2}} \circ \rho \circ (\sigma_X)_* \,. \end{array} \tag{5.2.20}$$

Proposition 5.2.5 The following diagram commutes

$$\begin{tikzcd} \mathcal{S}(0,\mathcal{C}) \arrow[r, "\text{homeo.}"] \arrow[d, "i_{\mathcal{S}(0)}(\sigma_X,\sigma_G)^{\pm}"'] & \mathcal{R}(\mathcal{C}) \arrow[d, "i_{\mathcal{R}(\mathcal{C})}(\sigma_X,\sigma_G)^{\pm}"] \\ \mathcal{S}(0,\mathcal{C}) \arrow[r, "\text{homeo.}"] & \mathcal{R}(\mathcal{C})\,. \end{tikzcd}$$

Proof: Outside S, Proposition 5.2.5 is true thanks to Proposition 3.5.1 and Proposition 3.5.2 . Since $\beta = 0$, then $L_i = G$, for all $1 \leq i \leq r$. Let $D \in \mathcal{S}(0)$ with $\mu(D) = \mathcal{C} \in \mathrm{Conj}(G)$. Let $c_i = \mathrm{hol}(D)$ be the loop enclosing x_i such that $\rho([c_i]) = \mathcal{C}_i$ and let $c'_{\sigma(i)} := \mathrm{hol}(i_{\mathcal{S}(0)}(\sigma_X,\sigma_G)^{\pm}(D))$. Then $c'_{\sigma(i)} = i_{\mathcal{R}(\mathcal{C})}(\sigma_X,\sigma_G)^{\pm}(c_i)$ and this is a loop enclosing $x_{\sigma(i)}$ with $\rho([c'_{\sigma(i)}]) = \mathcal{C}_{\sigma(i)}$. $\square$

5.3 Real parabolic G-Higgs bundles

Let G be a real form a reductive connected Lie group $G^{\mathbb{C}}$, and σ_G be a (μ,τ)-compatible conjugation of $G^{\mathbb{C}}$ as in the previous section. Let (X,σ_X) be a connected compact Klein surface and let $S = \{x_1,\ldots,x_r\}$ be a finite real set of points of X. We fix a maximal torus T and an alcove $\mathcal{A} \in \mathfrak{t}$ invariant by σ_G. (See Proposition 5.2.1). From Proposition 1.3.5, it follows that $d\theta := d(\sigma \circ \tau)$ is a Cartan involution for Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the Cartan decomposition. Since $d\sigma_G$ commutes with $d\theta$ then $d\sigma_G$ preserves $\mathfrak{h}$ and $\mathfrak{m}$ and by abuse of notation we denote $d\sigma_G$ to the restriction to this spaces. The isotropy representation $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$ defined on (3.1.1) is a $(\sigma_G, d\sigma_G)$-real representation. Let $Z(H^{\mathbb{C}})$ be the center of $H^{\mathbb{C}}$ and $Z(H^{\mathbb{C}})_2^{\sigma_G}$ be the subgroup of elements of $Z(H^{\mathbb{C}})$ of order two that are invariants under σ_G.

Definition 5.3.1 Let $\alpha = (\alpha_1, \ldots, \alpha_r) \in \prod_{i=1}^{r} \sqrt{-1}\,\bar{\mathcal{A}}$ be a (S, σ_G)-real element and $c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap \ker \iota = Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})$. A $(\sigma_X, \sigma_G, c, \alpha, \pm)$-**real parabolic structure** on a parabolic G-Higgs bundle with parabolic structure $(Q_1, \ldots, Q_r)$ of weight α is a (σ_X, σ_G, c)-real structure σ_E on E, such that

(1) $(Q_1, \ldots, Q_r)$ is $(S, \mathrm{Ad}(f))$-compatible, where $\mathrm{Ad}(f)$ is defined in (5.2.10)

(2) $P\mathrm{Ad}(f)(\varphi) = \pm\varphi$, where $P\mathrm{Ad}(f)$ is defined in (5.2.14).

Definition 5.3.2 A $(\sigma_X, \sigma_G, c, \alpha, \pm)$-**real parabolic G-Higgs bundle** is a triple (E, φ, σ_E), where (E, φ) is a parabolic G-Higgs bundle and σ_E is a $(\sigma_X, \sigma_G, c, \alpha, \pm)$-real parabolic structure.

Remark 5.3.1 There is an equivalent definition of real parabolic G-Higgs bundle using Proposition 1.6.1: A $(\sigma_X, \sigma_G, c, \alpha, \pm)$-real parabolic bundle is a triple (E, φ, f) where (E, φ) is a parabolic Higgs bundle and $f : E \to \sigma_X^* \sigma_G E$ is an isomorphism of G-bundles such that $\sigma_X^* \sigma_G \phi \circ \phi = c\,\mathrm{Id}_E$, and f satisfies (1) and (2) of Definition 5.3.1.

Example 5.3.1 (Real parabolic vector bundles) Let $H = \mathrm{U}(n)$, $H^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$ and σ_G be the complex conjugation of matrices. Let $\mathbb{V} = \mathbb{C}^n$ and $\sigma_{\mathbb{V}} = \mathrm{conj}$ be the complex conjugation. The standard representation is $(\sigma_G, \sigma_{\mathbb{V}})$-real. This representation associates to a real parabolic G-Higgs bundle (E, σ_E, φ) a real parabolic Higgs vector bundle (V, σ_V, φ) that is a real Higgs vector bundle equipped with a strictly decresing flag over each x_i preserved by σ_V. (See [3]).

Example 5.3.2 (Quaternionic parabolic vector bundles) Let $H = \mathrm{U}(n)$, $H^{\mathbb{C}} = \mathrm{GL}(2m, \mathbb{C})$ and $\sigma_G(A) = \mathrm{J}_m \circ \bar{A} \circ \mathrm{J}_m^{-1}$, for any $A \in \mathrm{GL}(n, \mathbb{C})$. ($J_m$ is defined in Example 1.2.2). Let $\mathbb{V} = \mathbb{C}^{2m}$ and $\sigma_{\mathbb{V}} v := \mathrm{J}_m \bar{v}$, for every $v \in \mathbb{V}$. The standard representation is $(\sigma_G, \sigma_{\mathbb{V}})$-quaternionic. (See Example 1.7.4). This representation associates to a real parabolic G-Higgs bundle (E, σ_E, φ) a quaternionic parabolic Higgs vector bundle (V, σ_V, φ) that is a real Higgs vector bundle equipped with a strictly decresing flag over each x_i preserved by σ_V .

Example 5.3.3 (Real unitary parabolic vector bundles) Let $H = \mathrm{U}(n)$, $H^{\mathbb{C}} = \mathrm{GL}(n,\mathbb{C})$ and $\sigma_G(A) = \mathrm{U}_{p,q}(\bar{A}^T)^{-1}\mathrm{U}_{p,q}$, for any $A \in \mathrm{GL}(n,\mathbb{C})$. ($\mathbf{i}_{p,q}$ is defined in Example 1.2.4) Let $\mathbb{V} = \mathbb{C}^n$ and $\sigma_{\mathbb{V}}$ the real unitary structure given in Example 1.2.4. The standard representation is $(\sigma_G, \sigma_{\mathbb{V}})$-real. (See Example 1.7.5). It associates to a real parabolic G-Higgs bundle (E, σ_E, φ) a real unitary parabolic Higgs vector bundle (V, θ_V, φ) that is a real Higgs vector bundle equipped with a strictly decresing flag

$$V_x \supset F^1V_x \supset F^2V_x \supset \ldots \supset F^{k_x+1}V_x = 0,$$

of linear subspaces over each x_i such that for every $1 \leq i \leq k_x + 1$,

$$\theta_V(F^iV(x)) = F^{k_x+1-i}V(\sigma_X(x)) \, .$$

5.4 Stability conditions

Stability notions for real parabolic G-Higgs bundles are almost the same that in parabolic case except that now, we put extra reality conditions on reductions: we only take reductions of the structure group of E from G to P_s such that the reduced bundle E_{P_s} is preserved under $\sigma_{\mathrm{Ad}(E)}$ defined in Example 1.7.2. Let us see the definitions in more detail.

Definition 5.4.1 Let $z \in \sqrt{-1}\mathfrak{z}$. A $(\sigma_X, \sigma_G, c, \alpha, \pm)$-real parabolic G-Higgs bundle (E, φ, σ_E) is:

- z-**semistable** if for every reduction of E to P_s such that

  $$\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}(E_{P_s}) = \mathrm{Ad}(E_{P_s})$$

  and $\varphi|_{X\backslash D} \in H^0(X\backslash S, E_{P_s}(\mathfrak{m}_s) \otimes K)$, then

  $$\mathrm{pardeg}_\alpha(E)(s,\sigma) - \langle z, s\rangle \geq 0 \, ,$$

- z-**stable** if we restrict $s \in \sqrt{-1}\mathfrak{h}_i$, then for every reduction of E to P_s such that

  $$\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}(E_{P_s}) = \mathrm{Ad}(E_{P_s})$$

and $\varphi|_{X\backslash D} \in H^0(X\backslash S, E_{P_s}(\mathfrak{m}_s) \otimes K)$, we have

$$\mathrm{pardeg}_\alpha(E)(s,\sigma) - \langle z, s\rangle > 0\,,$$

- z-**polystable** if it is c-semistable and if equality occurs for some $s \in \sqrt{-1}\mathfrak{h}_i$ and σ then there is $g \in PE(H^{\mathbb{C}})$ such that $g(E,\varphi) = (E',\varphi)$ admits a holomorphic reduction σ_{L_s} to L_s such that:

  (1) $\sigma_{\mathrm{Ad}(E)}\mathrm{Ad}(E_{L_s}) = \mathrm{Ad}(E_{L_s})$

  (2) $\varphi|_{X\backslash D} \in H^0(X\backslash D, E_{L_s}(\mathfrak{m}_s^0) \otimes K)$

  (3) E_{L_s} has a parabolic structure such that E' is induced from E_{L_s} through the injection $L_s \subset H^{\mathbb{C}}$.

Remark 5.4.1 Let (E,φ,σ_E) be a $(\sigma_X,\sigma_G,c,\alpha,\pm)$-real parabolic G-Higgs bundle. If the underlying parabolic G-Higgs bundle (E,φ) is z-(semi)stable or z-polystable then (E,φ,σ_E) is z-(semi)stable or z-polystable, respectively.

Definition 5.4.2 Two real-parabolic bundles $(E_0,\sigma_{E_0},\{Q_{0,i}\})$ and $(E_1,\sigma_{E_1},\{Q_{1,i}\})$ are **real meromorphically equivalent** if there is an $(\sigma_{E_0},\sigma_{E_1})$-real isomorphism of principal bundles

$$\psi : E_0|_{X^*} \to E_1|_{X^*}\,,$$

such that

$$\sigma_{\mathcal{G}_\alpha} f(\psi) = f(\sigma_{E_2} \circ \psi \circ \sigma_{E_1})\,,$$

where $f(\psi)$ is defined in 5.1.5 and $\sigma_{\mathcal{G}_\alpha}$ is an involution of sheaves induced by σ_X and σ_G.

We denote by $\mathcal{M}_z(\sigma_X,\sigma_G,\alpha,\pm)$ the moduli space of real meromorphically equivalent classes of $(\sigma_X,\sigma_G,c,\alpha,\pm)$-real polystable parabolic Higgs bundles.

Let $\mathcal{L} := (\mathcal{L}_1,\ldots,\mathcal{L}_r) \in \prod_{i=1}^r \widetilde{\mathfrak{m}_i^0}/\widetilde{L}_i$ be $(s,\mathrm{Ad}(f))$-real element. The moduli space $\mathcal{M}_z(\sigma_X,\sigma_G,\alpha,\pm,\mathcal{L}) := \rho^{-1}(\mathcal{L})$.

We denote by $\mathcal{M}_z^*(\sigma_X,\sigma_G,\alpha,\pm,\mathcal{L})$ to the moduli space of simple, stable and with vanishing obstruction class.

5.5 Hitchin-Kobayashi correspondence

First, we recall from [22, Definition 2.10] the notion of α-adapted metric and the construction of the initial metric h_0, which is described in [22, p.22].

Definition 5.5.1 Let (E, φ) be a parabolic Higgs bundle of weight $\alpha = (\alpha_1, \ldots, \alpha_r)$. A metric h on E is α**-adapted** if for any parabolic point x_i in the trivialization e_i there is a meromorphic gauge transformation $g \in PE(H^{\mathbb{C}})$ near x_i, such that in the trivialization $g(e_i)$ has the form

$$h = h_0 \cdot |z|^{-\alpha_i} e^c , \tag{5.5.21}$$

where h_0 is the standard constant metric, $\mathrm{Ad}(|z|^{-\alpha_i})c = o(\log |z|)$, $\mathrm{Ad}(|z|^{-\alpha_i})dc \in L^2$ and $\mathrm{Ad}(|z|^{-\alpha_i})F_h \in L^1$.

Definition 5.5.2 If we descompose $\mathrm{Gr}\,\mathrm{Res}_{x_i}\,\varphi = s_i + Y_i$ into its semisimple and nilpotent parts. Then, we complete Y_i into a Konstant-Rallis $\mathfrak{sl}_2$-triple (H_i, X_i, Y_i). The α-adapted **model metric** is given near a parabolic point x_i by

$$h_0 = |z|^{-2\alpha_i}(-ln|z|)^{H_i} . \tag{5.5.22}$$

Definition 5.5.3 Let (E, φ, σ_E) be a $(\sigma_G, \sigma_X, c, \alpha, \pm)$-real parabolic G-Higgs bundle. An α-adapted metric is σ_E**-compatible** if outside of the prescribed points the metric h satisfies that, if we denote by E_h the reduced H-bundle induced by h, $\sigma_E(E_h|_{X\setminus D}) = E_h|_{X\setminus D}$ and near any parabolic point x_i, there is a meromorphic gauge transformation $g \in PE(H^{\mathbb{C}})$ such that in the trivializations $g(e)$ and $\sigma_{PE(H^{\mathbb{C}})}g(\sigma_E e)$ one has

$$\sigma_{E(G/H)} \circ h \circ (\sigma_X)_* = h ,$$

where $\sigma_{E(G/H)}$ and $\sigma_{PE(H^{\mathbb{C}})}$ are the antiholomorphic involutions on $E(G/H)$ and $PE(H^{\mathbb{C}})$, respectively induced by σ_E and σ_G.

Proposition 5.5.1 The model metric h_0 is σ_E-compatible.

Proof: Since $\mathrm{Gr}\,\mathrm{Res}_{x_{\sigma(i)}}\,\varphi = s_{\sigma(i)} + Y_{\sigma(i)}$, we complete $Y_{\sigma(i)}$ into a Konstant-Rallis $\mathfrak{sl}_2$-triple $(H_{\sigma(i)}, X_{\sigma(i)}, Y_{\sigma(i)})$. Near $x_{\sigma(i)}$, in the trivialization $g(\sigma_G e)$ one has

$$h_0 = |\sigma(z)|^{-2\alpha_{\sigma(i)}}(-\ln|\sigma(z)|)^{H_{\sigma(i)}} .$$

But this is precisely $(\sigma_{E(G/H)} \circ h_0 \circ (\sigma_X)_*)$ near x_i in the trivialization e_i. $\square$

Definition 5.5.4 Let $z \in \sqrt{-1}\mathfrak{z}$. A metric $h \in \Gamma(X \setminus D, E(G/H))$ is z-Hermite-Einstein-Higgs if satisfies

$$\Lambda(F_h + [\varphi, \tau_h \varphi]) = -\sqrt{-1}z\,,$$

where F_h is the curvature of the unique h-connection on E_H compatible with the holomorphic structure of E, τ_h is defined in (3.2.3) and Λ is the contraction with a Kähler form ω.

Theorem 5.5.5 (Hitchin-Kobayashi correspondence) Let $z \in \sqrt{-1}\mathfrak{z}$ and let (E, φ, σ_E) be a $(\sigma_G, \sigma_X, c, \alpha, \pm)$-real parabolic G-Higgs bundle equipped with the model metric h_0. Suppose that $\mathrm{pardeg}_\alpha(E)(s_\chi, \sigma) = \chi(z)$, for every character χ of $\mathfrak{g}$, where σ_0 is the unique reduction of the structure group when $P = H^{\mathbb{C}}$. Then (E, φ, σ_E) admits a σ_E-compatible z-Hermite-Einstein metric h, α-adapted and quasi-isometric to h_0 if and only if it is z-polystable.

Proof: If (E, φ, σ_E) admits a σ_E-compatible z-Hermite-Einstein metric h, α-adapted and quasi-isometric to h_0, then by [22, Theorem 5.1] the underlying parabolic bundle (E, φ) is z-polystable and this imply that (E, φ, σ_E) is z-polystable by Remark 5.4.1.

Conversely, let (E, φ, σ_E) be a z-polystable $(\sigma_G, \sigma_X, c, \alpha, \pm)$-real parabolic G-Higgs bundle. There is a real Jordan-Hölder reduction following the same steps that in Section 2.7, so we can suppose that (E, φ, σ_E) is z-stable. Near the parabolic points, the model metric is a Poincaré metric of the form

$$ds^2 = \frac{|dz|^2}{|z|^2 \ln^2 |z|^2}\,. \tag{5.5.23}$$

If we change coordinates $z = |z|e^{\sqrt{-1}\theta}$ and $t = \ln(-\ln|z|^2)$, then $ds^2 = dt^2 + e^{-2t}d\theta^2$. Consider the weighted C^0 and L^p spaces:

$$C^0_\delta = e^{-\delta t} C^0\,, \qquad L^p_\delta = e^{-(\delta + \frac{1}{p})t} L^p\,.$$

Let ∇^+ be the H-connection induced by h_0 on E and $\nabla = \nabla^+ \oplus \mathrm{ad}(\varphi)$. The **weighted spaces** C^k_δ (resp. the **weighted Sobolev spaces** $L^{k,p}_\delta$) of sections f of $\mathbb{E}(\mathfrak{g}^{\mathbb{C}})$ such that $\nabla^j f \in C^0_\delta$, (resp. L^p_δ), for $j \leq k$.

The space of metric is for a small $\delta > 0$ and a large p is given by

$$\mathcal{H} := \{h = h_0 e^s,\ s \in \hat{L}^{2,p}_{\delta}(\mathbb{E}(\sqrt{-1}\mathfrak{h}))\}\,. \tag{5.5.24}$$

Definition 5.5.6 The **Donaldson functional** of $s \in \hat{L}^{2,p}_{\delta}(\mathbb{E}(\sqrt{-1}\mathfrak{h}))$ is

$$M(h_0, h_0 e^s) := \sqrt{-1}\int_X \mathrm{Tr}(s\Lambda F(h_0)) + \int_0^1 (1-t)\|e^{\sqrt{-1}ts/2} D''(s) e^{-\sqrt{-1}ts/2}\|^2_{L^2(X)}\, dt\,,$$

where $D''(s) = \overline{\partial}^E(s) + [\varphi, s]\,.$

Definition 5.5.7 Take some big B. We define

$$\mathcal{S}^{\infty}(B) = \{s \in \hat{C}^{\infty}_{\delta}(\mathbf{E}(\sqrt{-1}\mathfrak{h})),\ \|F(h_0 e^s)\|_{L^{2,p}_{\delta}} \leq B\}\,.$$

We want to minimize M in $\mathcal{S}^{\infty}_{\mathbb{R}}(B)$ such that the metric that we obtain satisfies $\sigma_E E_h = E_h$. This is equivalent to minimize M in

$$\mathcal{S}^{\infty}_{\mathbb{R}}(B) = \{s \in \mathcal{S}^{\infty}(B), s \text{ is a real section}\}\,,$$

where the reality of the section is with respect to an antiholomorphic involution on $\mathcal{S}^{\infty}(B)$ induced by $\sigma_{\mathrm{ad}(E)}$. Assume that do not exist any constants C, C' such that

$$\sup|s| \leq C + C' M(h_0, h_0 e^s) \text{ for any } s \in \mathcal{S}^{\infty}_{\mathbb{R}}(B)\,.$$

Since the curvature is bounded, it is suffice to prove that

$$\|s\|_{L^1} \leq C + C' M(h_0, h_0 e^s)\,. \tag{5.5.25}$$

If there are no constants C, C' satisfying (5.5.25), then there is a subsequence $\{s_i\} \subset \mathcal{S}^{\infty}_{\mathbb{R}}(B)$ and positive real numbers $C_i \to \infty$ such that

$$|s_i|_{L^1} \to \infty,\ \ C_i M(h_0, h_0 e^{s_i}) \leq |s_i|_{L^1} =: l_i\,.$$

Let $u_i := l_i^{-1} s_i$, so that $\|u_i\|_{L^1} = 1$. Up to passing to a subsequence, the sections u_i converge (weakly and locally in $L^{1,2}$) to a nonzero and locally $L^{1,2}$ to a real section u_∞ of $\mathbf{E}(\sqrt{-1}\mathfrak{h})|_{X\setminus D}$. This section satisfies the following properties:

- the L^2 norm of $D'' u_\infty$ is finite,

- there exist real numbers $l_1 < \cdots < l_k$ and locally $L^{1,2}$ sections $\pi_1, \dots, \pi_k$ of $\mathrm{End}(E(\mathfrak{h}^{\mathbb{C}}))_{X\setminus D}$ such that $\mathrm{ad}(u_\infty) = \sum l_j \pi_j$,
- for each j let $\Pi^j = \pi_1 + \cdots + \pi_j$. Then $(1 - \Pi^j) D'' \Pi^j = 0$.

The regularity theorem of Uhlenbeck and Yau for locally $L^{1,2}$ we have that The section u_∞ is smooth by the Uhlenbeck and Yau theorem of $\mathbf{E}(\sqrt{-1}\mathfrak{h})|_{X\setminus D} \subset E(\mathfrak{h}^{\mathbb{C}})|_{X\setminus D}$. The section $\iota(u_\infty)$ defines a filtration on $E(\mathfrak{g})$ compatible with σ_{ad}. This defines a holomorphic σ_E-compatible reduction σ of the structure group of $E|_{X\setminus D}$ to the parabolic subgroup $P = P_s$, where $u_\infty(x) = s \in \sqrt{-1}\mathfrak{h}$ in the trivialization e .

Following the proof of [22, Lemma 5.3] the reduction extends to the parabolic points, giving a σ_E-compatible α-adapted metric.

The section u_∞ satisfies

$$
\begin{aligned}
2\pi \mathrm{pardeg}_\alpha(E)(\sigma, s) &= \int_{X\setminus D} \langle \Lambda F(h_0), u_\infty \rangle + \int_{X\setminus D} \langle \varpi(u_\infty)(\overline{\partial} u_\infty), \overline{\partial} u_\infty \rangle \leq \\
&\leq \int_{X\setminus D} \langle \Lambda F(h_0), u_\infty \rangle + \int_{X\setminus D} \langle \varpi(u_\infty)(D'' u_\infty), D'' u_\infty \rangle \leq 0 ,
\end{aligned}
$$

where the function $\varpi : \mathbb{R} \to \mathbb{R}$ is defined by $\varpi(0) = 0$ and as $\varpi(x) = x^{-1}$ if $x \neq 0$. We obtain a contradiction because by stability $2\pi \mathrm{pardeg}_\alpha(E)(\sigma, s) > 0$.

Therefore, there are constants C and C' satisfying (5.5.25). Taking a minimizing sequence $h_0 e^{\sqrt{-1} s_i}$ for the Donaldson funcional M, with $s_i \in \mathcal{S}^\infty_{\mathbb{R}}(B)$, after passing to a subsequence they converge weakly to s that belongs also to $\mathcal{S}^\infty_{\mathbb{R}}(B)$. Then, the metric $He^{\sqrt{-1}s}$ minimize M and it is a solution to the main theorem. □

We define a **forgetfull map**

$$
\mathcal{M}(\sigma_X, \sigma_G, c, \alpha, \pm) \to \mathcal{M}(\alpha) ,
$$

sending (E, σ_E, φ) to (E, φ). We denote by $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm)$ the image of the forgetfull map, by $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L})$ the the intersection of $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm)$ and $\mathcal{M}(\alpha, \mathcal{L})$. We denote by $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm)_{\mathrm{sm}}$ the intersection of $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm)$ and $\mathcal{M}(\alpha)_{\mathrm{sm}}$ and by $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L})_{\mathrm{sm}}$ the intersection of $\widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L})$ and $\mathcal{M}(\alpha)_{\mathrm{sm}}$.

5.6 Compatible parabolic filtered local systems

Let G be a real form a reductive connected Lie group $G^{\mathbb{C}}$, and σ_G be a (μ, τ)-compatible conjugation of $G^{\mathbb{C}}$ as in Section 5.2. Let (X, σ_X) be a connected compact Klein surface and let $S = \{x_1, \dots, x_r\}$ be a finite real set of points of X.

Definition 5.6.1 Let $\beta = (\beta_1, \dots, \beta_r) \in \prod_{i=1}^r \mathfrak{t}$ be a (S, σ_G)-compatible element and $c \in Z_2^{\sigma_G} \cap \ker \iota$.

A $(\sigma_X, \sigma_G, c, \beta, \pm)$-**compatible parabolic local system** is a triple (F, S, σ_F) such that $F_{X \setminus S}$ is a $(\sigma_X, \sigma_G, c, \pm)$ compatible local system (Definition 3.3.2) and for every x_i and every ray ρ_i going to x_i we have to choose a pair (P_i, χ_i) where

(1) P_i is a parabolic subgroup of $F(G)|_{\rho_i}$ isomorphic to P_{β_i}

(2) P_i is invariant under the monodromy transformation around x_i

(3) σ_F and σ_G induce an involution $\sigma_{F(G)}$ on $F(G)$ such that $(P_1, \dots, P_r) \in \prod_{i=1}^r F(G)|_{\rho_i}$ is a $(S, \sigma_{F(G)})$-compatible element

(4) χ_i is a stricly antidominant character of P_i

Definition 5.6.2 Let $\zeta \in \mathfrak{m}^H$. Let $Q \subset G$ be a parabolic subgroup of G and χ be an antidominant character of its Lie algebra $\mathfrak{q}$. We say that a $(\sigma_X, \sigma_G, c, \beta, \pm)$-compatible parabolic local system (F, D, σ_F) is:

- ζ-**semistable** if for any reduction F_Q of the local system invariant under the flat connection such that $\sigma_{\mathrm{Ad}(E)} \mathrm{Ad} F_Q = \mathrm{Ad} F_Q$, then

$$\operatorname{pardeg}(F)(Q, \chi, \sigma) := -\sum_i \deg\big((P_i, s_{\chi_i}), (Q, s_\chi)\big) - \langle \zeta, s_\chi \rangle \geq 0\,, \quad (5.6.26)$$

- ζ-**stable** if the inequality is strict for any non-trivial reduction,

- ζ-**polystable** if it is semistable and equality happens in (5.6.26) if and only if there is a reduction of the local system to a Levi subgroup $L \subset Q$ such that

$$\sigma_{\mathrm{Ad}(E)} \mathrm{Ad} F_L = \mathrm{Ad} F_L\,,$$

where L is a local system which induces F via the inclusion $L \subset G$.

We denote by $\mathcal{S}_\zeta(\sigma_X, \sigma_G, c, \beta, \pm)$ the moduli space of ζ-polystable $(\sigma_X, \sigma_G, c, \beta, \pm)$-compatible parabolic local systems.

5.7 Hitchin-Simpson correspondence

Theorem 5.7.1 (Hitchin-Simpson correspondence) Let (F, D, σ_F) be a polystable $(\sigma_X, \sigma_G, c, \beta, \pm)$-compatible parabolic local system, with vanishing parabolic degree with respect to all characters of $\mathfrak{g}$. It admits a σ_F-compatible harmonic metric h (compatible with the parabolic structure near the marked points), which induces a $(\sigma_G, \sigma_X, c, \alpha, \pm)$-real parabolic G-Higgs bundle, where α and β are related as in Table 5.1.

Proof: The parabolic local system (F, D) comes from a parabolic G-Higgs bundle (E, φ), where $F = \mathbb{E}$ and $D = A_h + \varphi - \tau_h \varphi$, with A_h being the unique h-connection compatible with the holomorphic structure of E. Near a parabolic point x_i, on a local frame e_i ,

$$\begin{aligned} D = d &+ \big(s_i - \tau(s_i) - \mathrm{Ad}(e^{\sqrt{-1}\theta\alpha_i})\frac{Y_i + X_i}{\ln r^2}\big)\frac{dr}{r} \\ &+ \sqrt{-1}\big(-\alpha_i + s_i + \tau(s_i) - \mathrm{Ad}(e^{\sqrt{-1}\theta\alpha_i})\frac{Y_i - H_i - X_i}{\ln r^2}\big)d\theta\,. \end{aligned} \tag{5.7.27}$$

Consider the initial metric

$$h_0 = r^{2(-s_i+\tau(s_i))}(-\ln r^2)^{\mathrm{Ad}(\sqrt{-1}\theta\alpha_i)(X_i+Y_i)}\,. \tag{5.7.28}$$

It is σ_F-compatible in the sense that: $\sigma_{F(G/H)} \circ h \circ (\sigma_X)_* = h$, where $\sigma_{F(G/H)}$ is the antiholomorphic involution in $F(G/H)$ induced by σ_F and σ_G. The functional space of σ_F-compatible metrics is

$$\mathcal{H}^\infty_\mathbb{R} = \{h_0\, e^s,\ s \in \hat{C}^\infty_\delta(\mathfrak{m}) \text{ and } s \text{ is real}\}\,, \tag{5.7.29}$$

where the reality of the section s is with respect to $\sigma_{\mathrm{ad}(F))}$ induced by σ_F and σ_G. We can suppose doing a Jordan-Hölder reduction that (F, D, σ_F) is stable. We are looking for $h \in \mathcal{H}^\infty_\mathbb{R}$ minimizing

$$\int_X (|\psi_h|^2 - |\psi_{h_0}|^2)$$

on $\mathfrak{h}$, under the constraint $\|(D_h^+)^*\psi_h\|_{L^p_\delta} \leq B$. The proof is similar to the Hitchin-Kobayashi correspondence: we choose a minimizing sequence of σ_F-compatible metrics, if they do not converge, there is a reduction and we obtain a contradiction with stability. The minimum of the sequence is a metric that belongs to $\mathcal{H}^\infty_\mathbb{R}$ and it is σ_F-compatible harmonic metric.

The $(1,0)$-part of h defines the Higgs field φ. The σ_F-compatibility of h implies that φ is a real section, out of S. We can see that the extension constructed in [22, Proposition 6.6] is σ_E-compatible because in the gauge

$$e = f(-\ln r^2)^{\mathrm{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i/2} r^{-\alpha_i}\,,$$

one has

$$\varphi = \mathrm{Ad}\big(r^{\alpha_i}(-\ln r^2)^{-\mathrm{Ad}(e^{\sqrt{-1}\theta\alpha_i})H_i/2}\big)(\varphi_0 + b')\,.$$

and then $\sigma_{E(G/H)} \circ \varphi \circ (\sigma_X)_* = \varphi$. $\square$

We define a **forgetfull map**

$$\mathcal{S}(\sigma_X, \sigma_G, c, \beta, \pm) \to \mathcal{S}(\beta)\,,$$

sending (F, S, σ_F) to F. We denote by $\widetilde{\mathcal{S}}(\sigma_X, \sigma_G, c, \beta, \pm)$ the image of the forgetfull map. The map μ defined in (5.1.9) restricts to

$$\mu : \mathcal{S}_\zeta(\sigma_X, \sigma_G, c, \beta, \pm) \longrightarrow \prod_i \mathrm{Conj}(L_{\beta_i})\,.$$

Let $\mathcal{C} = (\mathcal{C}_1, \cdots, \mathcal{C}_r) \in \prod_{i=1}^r \mathrm{Conj}(L_{\beta_i})$ be a (S, σ_G)-compatible element. We define

$$\mathcal{S}(\sigma_X, \sigma_G, c, \beta, \pm, \mathcal{C}) := \mu^{-1}(\mathcal{C})\,.$$

We denote by $\widetilde{\mathcal{S}}(\sigma_X, \sigma_G, c, \beta, \pm, \mathcal{C})$ the image of the forgetfull map and we denote by $\widetilde{\mathcal{S}}(\sigma_X, \sigma_G, c, \beta, \pm, \mathcal{C})$ the intersection of $\widetilde{\mathcal{S}}(\sigma_X, \sigma_G, c, \beta, \pm)$ and $\mathcal{S}(\beta, \mathcal{C})$. We denote by $\widetilde{\mathcal{S}}(\sigma_X, \sigma_G, c, \beta, \pm, \mathcal{C})_{\mathrm{sm}}$ the intersection of $\widetilde{\mathcal{S}}(\sigma_X, \sigma_G, c, \beta, \pm, \mathcal{C})$ and $\mathcal{S}(\beta, \mathcal{C})_{\mathrm{sm}}$.

Definition 5.7.2 Let $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_r\} \in \prod_{i=1}^r \mathrm{Conj}(G)$ a (S, σ_G)-compatible element. A $\mathcal{C}$-**representation of** $\Gamma(X \setminus D)$ **in** G is a $(\sigma_X, \sigma_G, c, \pm)$-compatible representation (Definition 1.4.3) such that if we restrict to $\pi_1(X)$, then the image of any class of a loop around x_i is $\mathcal{C}_i$, for $1 \leq i \leq r$.

We denote by $\mathcal{R}(\sigma_X, \sigma_G, c, \mathcal{C})$ the moduli space of reductive $\mathcal{C}$-representations. There is a **forgetfull map**

$$\mathcal{R}(\sigma_X, \sigma_G, c, \pm, \mathcal{C}) \to \mathcal{R}(\mathcal{C})\,.$$

We denote $\widetilde{\mathcal{R}}(\sigma_X, \sigma_G, c, \pm, \mathcal{C})$ to the image of the forgetfull map and by $\widetilde{\mathcal{R}}(\sigma_X, \sigma_G, c, \pm, \mathcal{C})_{\mathrm{sm}}$ the intersection of $\widetilde{\mathcal{R}}(\sigma_X, \sigma_G, c, \pm, \mathcal{C})$ and $\mathcal{R}(\mathcal{C})_{\mathrm{sm}}$.

Theorem 5.7.3 [Non-abelian Hodge correspondence] If the weights and the monodromies are related by Table 5.1 and they are (S, σ_G)-compatible elements, then:

(1) $\mathcal{M}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L})$ is homeomorphic to $\mathcal{S}(\sigma_X, \sigma_G, c, \beta, \pm, \mathcal{C})$.

(2) $\mathcal{M}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L})_{\mathrm{sm}}$ is differomorphic to $\mathcal{S}(\sigma_X, \sigma_G, c, \beta, \pm, \mathcal{C})_{\mathrm{sm}}$.

(3) $\mathcal{S}(\sigma_X, \sigma_G, c, 0, \pm, \mathcal{C})$ is homeomorphic to $\mathcal{R}(\sigma_X, \sigma_G, c, \pm, \mathcal{C})$.

(4) $\mathcal{S}(\sigma_X, \sigma_G, c, 0, \pm, \mathcal{C})_{\mathrm{sm}}$ is diffeomorphic to $\mathcal{R}(\sigma_X, \sigma_G, c, \pm, \mathcal{C})_{\mathrm{sm}}$.

Proof: (1) and (2) are consequence of Theorem 5.7.1. From Theorem 3.3.6, it follows that (3) and (4) are true without prescribing $\mathcal{C} \in \mathrm{Conj}(G)$. Since the holonomies arround the parabolic points $\mathcal{C}$ satisfy that they are (S, σ_G)-compatible, the parabolic local system is compatible as well. $\square$

5.8 Description of the fixed points locus

Proposition 5.8.1 The fixed points of the involution $i_{\mathcal{M}(\alpha)}(\sigma_X, \sigma_G)^{\pm}$ and the moduli space of $(\sigma_X, \sigma_G, c, \alpha, \pm)$-real parabolic G-Higgs bundles of class $\mathcal{L}$ are related by:

(1)
$$\mathcal{M}(\alpha, \mathcal{L})^{i_{\mathcal{M}(\alpha)}(\sigma_X, \sigma_G)^{\pm}} \supseteq \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L})\,.$$

(2) For $g(X) \geq 2$ and if we restrict to the smooth locus sm, then

$$\mathcal{M}(\alpha, \mathcal{L})_{\mathrm{sm}}^{i_{\mathcal{M}(\alpha)}(\sigma_X, \sigma_G)^{\pm}} \cong \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{M}}(\sigma_X, \sigma_G, c, \alpha, \pm, \mathcal{L})_{\mathrm{sm}}\,.$$

Proof: The inclusion of (1) is a consequence of Remark 5.3.1

Let $(E,\varphi) \in \mathcal{M}(\alpha,\mathcal{L})^{i_{\mathcal{M}(\alpha)}(\sigma_X,\sigma_G)^{\pm}}$, then there is an isomorphism $f : E \to \sigma_X^*\sigma_G E$ such that $(f(E), \pm P\mathrm{Ad}(f)(\varphi)) \cong (E,\varphi)$, where $P\mathrm{Ad}(f)$ is defined in (5.2.14). The composition $\sigma_X^*\sigma_G f \circ f$ belongs to $\mathrm{Aut}(E,\varphi)$ that is equal to $Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota) = Z(H^{\mathbb{C}}) \cap Z(G^{\mathbb{C}})$ because (E,φ) is simple. (See Definition 5.1.12). Let $c := \sigma_X^*\sigma_G(f) \circ f \in Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\rho)$. Since f commutes with $\sigma_X^*\sigma_G f \circ f$, then $\sigma_G(c) = c$. From the equivalence of definitions of real parabolic G-Higgs bundles mentioned above, it follows that (E,φ,f) is a $(\sigma_X,\sigma_G,c,\alpha,\pm)$-real Higgs parabolic bundle. □

Proposition 5.8.2 The fixed points of the involution $i_{\mathcal{S}(\beta)}(\sigma_X,\sigma_G)^{\pm}$ and moduli space of $(\sigma_X,\sigma_G,c,\beta)$-compatible parabolic G-local systems of class $\mathcal{C}$ are related by:

(1)
$$\mathcal{S}(\beta,\mathcal{C})^{i_{\mathcal{S}(\beta)}(\sigma_X,\sigma_G)^{\pm}} \supseteq \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{S}}(\sigma_X,\sigma_G,c,\beta,\pm,\mathcal{C})\,.$$

(2) For $g(X) \geq 2$ and if we restrict to the smooth locus sm, then
$$\mathcal{S}(\beta,\mathcal{C})_{\mathrm{sm}}^{i_{\mathcal{S}(\beta)}(\sigma_X,\sigma_G)^{\pm}} \cong \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{S}}(\sigma_X,\sigma_G,c,\beta,\pm,\mathcal{C})_{\mathrm{sm}}\,.$$

Proof: It is a consequence of Proposition 5.8.1, Proposition 5.2.4 and the bijection of moduli spaces $\mathcal{M}(\sigma_X,\sigma_G,c,\alpha,\pm,\mathcal{L})$ and $\mathcal{S}(\sigma_X,\sigma_G,c,\beta,\pm,\mathcal{C})$ proved in Proposition 5.7.3. □

Proposition 5.8.3 The fixed points of the involution $i_{\mathcal{R}}(\sigma_X,\sigma_G)^{\pm}$ and moduli space of $(\sigma_X,\sigma_G,c,\mathcal{C})$-real representations of $\Gamma(X\backslash S)$ of class $\mathcal{C}$ are related by:

(1)
$$\mathcal{R}(\mathcal{C})^{i_{\mathcal{R}}(\sigma_X,\sigma_G)^{\pm}} \supseteq \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{R}}(\sigma_X,\sigma_G,c,\pm,\mathcal{C})\,.$$

(2) For $g(X) \geq 2$ and if we restrict to the smooth locus sm, then
$$\mathcal{R}(\mathcal{C})_{\mathrm{sm}}^{i_{\mathcal{R}}(\sigma_X,\sigma_G)^{\pm}} \cong \bigcup_{c \in Z(H^{\mathbb{C}})_2^{\sigma_G} \cap Z(G^{\mathbb{C}})} \widetilde{\mathcal{R}}(\sigma_X,\sigma_G,c,\pm,\mathcal{C})_{\mathrm{sm}}\,.$$

Proof: It is a consequence of Proposition 5.8.1, Proposition 5.2.5 and the bijection of moduli spaces $\mathcal{S}(\beta,0)$ and $\mathcal{R}(\beta)$ proved in Proposition 5.7.3. □

Appendix A

Parabolic degree

A.1 Spherical Tits buildings

A.1.1 Spherical Tits buildings

This is a review of [50, Sections 2 and 3] and [44, Chapter 3]. A Coxeter matrix M is an $n \times n$ symmetric matrix such that $M_{ii} = 1$ and $M_{ij} > 1$, for all $1 \leq i, j \leq n$. Let S be a set of n elements of a group. Given a Coxeter matrix $M = (M_{st})_{s,t \in S}$, the associated Coxeter group W is given by the presentation

$$W = \langle r_1, \ldots, r_n | (r_i r_j)^{M_{ij}} = 1 \rangle \ .$$

The pair (W, S) is called a finite Coxeter system associated to the Coxeter matrix M. Let $\mathbb{V}$ be a vector space with a basis $\{e_s\}_{s \in S}$ equipped with a bilinear form $B(e_s, e_t) = -\cos(\frac{\pi}{M_{st}})$. The Coxeter group W acts on $\mathbb{V}$ by

$$s(v) = v - 2\frac{B(e_s, v)}{B(e_s, e_s)} e_s \ .$$

A Coxeter system determines a set of chambers of $\mathbb{V}$ on which W acts simply transitively.

Definition A.1.1 The **Coxeter complex** $\Sigma(W, S)$ of a finite Coxeter system (W, S) is the intersection of the chambers with the unit sphere.

Definition A.1.2 A simplicial complex Δ is a **spherical Tits building** if it contains a family of subsets called apartments such that:

(1) Each apartment is a Coxeter complex.

(2) Any two simplex are contained in an apartment.

(3) If σ and σ' are two simplices of the apartments Σ and Σ', respectively, such that $\sigma, \sigma' \in \Sigma \cap \Sigma'$, then there is an isomorphism $\Sigma \cong \Sigma'$ fixing σ and σ' pointwise.

Given two simpleces σ_1, σ_2 of a spherical Tits building Δ, by Definition A.1.2 there is an apartment a such that $\sigma_1, \sigma_2 \in a$ and there is an isomorphism $\alpha : \Sigma(W, S) \to a$, with $\Sigma(W, S)$ being a Coxeter complex. Let σ_1^{Σ} be the element of $\Sigma(W, S)$ such that $\alpha(\sigma_1^{\Sigma}) = \sigma_1$. Let $W_{\sigma_1} := \mathrm{Stab}_W(\sigma_1^{\Sigma})$ be the stabilizer of σ_1^{Σ} by W.

Definition A.1.3 The **relative position** (σ_1, σ_2) of two simplices σ_1, σ_2 of a spherical Tits building Δ is an element in $W_{\sigma_1} \backslash W / W_{\sigma_2}$ such that

$$\alpha(\, (\sigma_1, \sigma_2)\, \sigma_1^{\Sigma}\,) = \sigma_2 \ .$$

A.1.2 The spherical Tits building $\Delta(G)$

Let G be a semisimple Lie group. Let $H \subset G$ be a maximal compact subgroup and let $\mathfrak{g}$ and $\mathfrak{h}$ be their respective Lie algebras. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the Cartan descomposition. Fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and let A be the group $\exp(\mathfrak{a})$. We define the simplicial complex $\Delta(G)$ as follows: the simpleces are proper parabolic subgroups of G. The vertices are maximal parabolic subgroups and a simplex $P \in \Delta(G)$ has vertices $P_1, \ldots, P_n$ if and only if $P = P_1 \cap \ldots \cap P_n$. Let g be an element of the group G. We define the apartment

$$\Sigma(g) = \{P \in \Delta(G),\ \ gAg^{-1} \subset P\} \ .$$

It is a Coxeter complex of the Coxeter system associated to the Weyl group $W := N_H(\mathfrak{a})/Z_H(\mathfrak{a})$.

Theorem A.1.4 (See [44], Theorem 3.15) The simplicial complex $\Delta(G)$ with the set of apartments $\{\Sigma(g)\}_{g \in G}$ is a spherical Tits building.

Remark A.1.1 In this case, the relative position of two simplices (See Definition A.1.3) is an element of $W_{\sigma_1}\backslash W/W_{\sigma_2} \cong P_{\sigma_1}\backslash G/P_{\sigma_2}$.

A.1.3 The spherical Tits building $\Delta(\partial_\infty(X))$

A Riemannian metric g on a smooth manifold X is a family of positive definite inner products in T_xX that varies smoothly as x varies in X. A Riemannian space (X, g) is a pair, where X is a smooth manifold and g is a Riemannian metric on X. A Riemannian symmetric space is a Riemannian space (X, g) such that for each point $x \in X$ there is an isometry of X fixing x and acting on T_xX by minus the Identity. A Riemannian symmetric space is of non compact type if its sectional curvature is non positive. Fix a Riemannian symmetric space of non compact type X. By Cartan classification of symmetric spaces, we can suppose, without loss of generality, that X is a homogeneous space $X = H\backslash G$, where G is a semisimple (complex or real) Lie group and $H \subset G$ is a maximal compact subgroup of G. Let d be the distance function of X associated to the Riemannian metric g. Two directed unit speed geodesics γ_1, γ_2 of X are equivalent if

$$\lim_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) < \infty \ .$$

Definition A.1.5 The **boundary at infinity** $\partial_\infty X$ **of** X is the set of equivalence classes of geodesics in X.

Definition A.1.6 The **Tits distance** $\angle_{\mathrm{Tits}}(\epsilon_1, \epsilon_1)$ of two points $\epsilon_1, \epsilon_1 \in \partial_\infty X$, that are equivalence classes of two geodesics γ_1, γ_2 is

$$\angle_{\mathrm{Tits}}(\epsilon_1, \epsilon_2) := \sup_{x\in X} \angle_x(\gamma_1, \gamma_2) \ ,$$

with $\angle_x(\gamma_1, \gamma_2)$ being the limit when $t \to 0$ of the angle between the segments $\overline{xy_t}$ and $\overline{xz_t}$, where $x = \gamma_1(0) = \gamma_2(0)$, $y_t = \gamma_1(t)$ and $z_t = \gamma_2(t)$. This definition does not depend on the choice of the representants of the classes ϵ_1, ϵ_2. (See [53, Section 2.2]).

Proposition A.1.1 Let σ_G be a real structure on G preserving H. Let g be a Riemannian metric σ_G-invariant and d be the associated distance, then σ_G defines an isometry in $\partial_\infty(H\backslash G)$ with respect to the Tits distance.

Proof: Is a consequence of the following formula given in [53, Section 2.2]

$$2 \sin \frac{\angle_{\mathrm{Tits}}(\epsilon_1, \epsilon_1)}{2} = \lim_{t \to \infty} \frac{d(\gamma_1(t), \gamma_2(t))}{t} \ .$$

□

Remark A.1.2 Fixing a base point $o = [H] \in X$ there is a bijective correspondence between the unit sphere on T_oX and $\partial_\infty X$ that consists of sending a unit vector v of T_oX to the geodesic from o with initial tangent vector v.

In the space $X \cup \partial_\infty X$ there is a topology such that: it restricts to the topology of X, it restricts to the topology of the unit sphere of T_oX and finally, if a sequence $\{x_j\}$ of elements in X converges to $[\gamma] \in \partial_\infty X$, then the set of directed geodesic from o to x_j converges to a geodesic equivalent to γ.

Definition A.1.7 The space $X \cup \partial_\infty X$ with this topology is the **geodesic compactificacion of** X.

As in the previous section, let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the Cartan descomposition and fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and let A be the group $\exp(\mathfrak{a})$. Let Σ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $W = N_H(\mathfrak{a})/Z_H(\mathfrak{a})$ be the Weyl group. The set of Weyl chambers of $\mathfrak{a}$ are the connected components of $\mathfrak{a} \setminus \cup_{\alpha \in \Sigma} \mathrm{Ker}(\alpha)$. Let us fix one $\mathfrak{a}^+$ to be the positive Weyl chamber. The positive roots are those that take positive values on $\mathfrak{a}^+$. A positive root is simple if it can not be writen as the sum of two positive roots. We denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ the set of simple roots.

Definition A.1.8 Let I be a proper subset of Δ, the **Weyl chamber face** C_I is the following subset of the clousure of $\mathfrak{a}^+$

$$C_I := \{H \in \overline{\mathfrak{a}^+}, \ \ \alpha_i(H) > 0 \ \leftrightarrow \ \ \alpha_i \notin I\} \ .$$

The set of all Weyl chamber faces C_I varying I is equal to $\partial_\infty \mathfrak{a}^+$. Let $\{k_1, \dots, k_{|W|}\}$ be a complete set of representatives of W. The Weyl group acts transitively on the set of Weyl chambers, so any Weyl chamber can be writen as $\mathrm{Ad}(k_i)\mathfrak{a}^+$, where $k_i \in W$.

Proposition A.1.2 ([44], Proposition 3.7) The set of all Weyl chamber faces $\mathrm{Ad}(k_i)C_I$ varying I, determines a triangulation of the unit sphere S in $\mathfrak{a}$. The set of $\sigma(k_i, I) := \mathrm{Ad}(k_i)C_I \cap S$ forms a simplicial complex, where $\sigma(k_i, I)$ is a face of $\sigma(k_j, J)$ if $\sigma(k_i, I) \subset \overline{\sigma(k_j, J)}$.

Definition A.1.9 Given a convex cone J on T_oX that contains no lines, let $\partial_\infty(J) \subset \partial_\infty(X)$ be the subset of corresponding unit vectors in J (See Remark [A.1.2]).

Definition A.1.10 **Weyl chamber faces at infinity** are sets of the form

$$\partial_\infty(\,\mathrm{Ad}(k_i)C_I\,) = \mathrm{Ad}(k_i)\partial_\infty(C_I) = k_i \cdot \partial_\infty(C_I)\ ,$$

where $\partial_\infty(C_I)$ is described in Definition A.1.9.

Let $\Delta(\partial_\infty(X))$ be the set of all Weil chamber faces at infinity. By Proposition [A.1.2], it is a simplicial complex.

Definition A.1.11 A flat F in X is a complete, totally geodesic flat submanifold of maximal dimension. Any flat of X can be writen as $\exp(Ad(h)\mathfrak{a}) = h\exp(\mathfrak{a})o$, for some $h \in H$. Given a flat F of X, the **associated apartment to** F is the subcomplex of $\Delta(\partial_\infty(X))$ contained in $\partial_\infty(F)$.

Proposition A.1.3 ([44], Proposition 3.20) The simplicial complex $\Delta(\partial_\infty(X))$, with the set of associated apartments to all the flats of X, is a spherical Tits building.

A.1.4 Equivalence of spherical Tits buildings

In the previous subsections, we have constructed two spherical Tits buildings $\Delta(G)$ and $\Delta(\partial_\infty(X))$. The relation between both is given by the following theorem.

Theorem A.1.12 ([44], Propositions 3.18 and 3.20) The map

$$\begin{array}{ccc} \Delta(\partial_\infty(X)) & \longrightarrow & \Delta(G) \\ k_i \cdot \partial_\infty(C_I) & \longmapsto & k_i P^I (k_i)^{-1}\ , \end{array}$$

is an isomorphism of spherical Tits buildings, where $k_i \in W = N_H(\mathfrak{a})/Z_H(\mathfrak{a})$. If $k_i \cdot \partial_\infty(C_I)$ is a face of $k_j \cdot \partial_\infty(C_J)$, then $k_j P^J (k_j)^{-1}$ is a face of $k_i P^I (k_i)^{-1}$.

Corollary A.1.13 Any parabolic subgroup $k_i P^I k_i^{-1}$ is equal to $\mathrm{Stab}_G(z)$, for every $z \in k_i \cdot \partial_\infty(C_I)$, .

Proof: Let z_1 and z_2 be two points in $\partial_\infty X$, by [44, Proposition 3.16], one has that $\mathrm{Stab}_G(z_1) = \mathrm{Stab}_G(z_2)$ if and only if z_1 and z_2 lie in the same Weyl chamber face at infinity $k_i \cdot \partial_\infty(C_I)$, but by Theorem [A.1.12] this Weyl chamber face corresponds to the parabolic subgroup $k_i P^I k_i^{-1}$. $\square$

A.2 Equivalence of definitions of pardeg

First, we recall from [22, Section B.2.], the definitions of relative degree and parabolic degree. Let $H\backslash G$ a symmetric space of non compact type and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the Cartan decomposition. Let $\mathcal{O}_H \subset \mathfrak{m}$ be an H-orbit in $\mathfrak{m}$. There is an identification

$$\begin{array}{rccc} \eta: & \mathcal{O}_H = H/(P_s \cap H) & \longrightarrow & P_s\backslash G \subset \partial_\infty(H\backslash G) \\ & s & \longmapsto & \lim_{t\to\infty} \exp(\sqrt{-1}ts)o \ , \end{array} \tag{A.2.1}$$

where o is the base point, fixed by H. Then $\mathcal{O}_H$ is a G-space. The action is given by $s \cdot g = s \cdot h = \mathrm{Ad}(h^{-1})s$, where $h \in H$ is given by the decomposition $G = H\, P_s$.

Definition A.2.1 Let B be the H-invariant inner product on $\mathfrak{h}$. The **relative degree** $\deg(s,s')$ **of** s and $s' \in \mathfrak{m}$ is given by

$$\deg(s,s') := \lim_{t\to\infty} B(\, s \cdot e^{-t\,s'}\,, s'\,)\ .$$

Let $T \subset H$ be a maximal torus and $\mathfrak{t}$ be its Lie algebra. We fix an alcove $\mathcal{A} \subset \mathfrak{t}$ containing $0 \in \mathfrak{t}$. Let $\bar{\mathcal{A}}$ be its clousure on $\mathfrak{t}$. Let $\alpha = \{\alpha_1, \dots, \alpha_r\}$ be a set of r elements of $\sqrt{-1}\bar{\mathcal{A}}$.

Definition A.2.2 (Biquard, Garcia-Prada and Mundet i Riera) The **parabolic degree** of a parabolic G-Higgs bundle E with parabolic structure $Q = \{Q_1, \dots, Q_r\}$ and weights α with respect to $s \in \sqrt{-1}\mathfrak{h}$ and a reduction σ of the structure group from G to a parabolic subgroup P_s is defined by the following sum

$$\mathrm{pardeg}_\alpha(E)(s,\sigma) := \deg E(s.\sigma) - \sum_i^r \deg\big((Q_i,\alpha_i),(E_\sigma(P_s)_{x_i}, s)\big)\ , \tag{A.2.2}$$

where χ is an antidominant character of the reduction $E_\sigma(P)_{x_i}$ and

$$\deg\big((Q_i,\alpha_i),(E_\sigma(P_s)_{x_i},s)\big) \tag{A.2.3}$$

is the relative degree of $\eta(\alpha_i)$ and $\eta(s)$. If $|\eta(s)| = |\eta(\alpha_i)| = 1$, then by [22, Proposition B.1.] the relative degree is equal to

$$\cos\angle_{\mathrm{Tits}}(\eta(\alpha_i),\eta(s)) \ ,$$

where $\angle_{\mathrm{Tits}}$ is the Tits distance on $\partial_\infty(H\backslash G)$.

Definition A.2.3 (Teleman and Woodward) In the case G is a semisimple Lie group.

$$\mathrm{pardeg}_\alpha(E)(s,\sigma) := \deg E(s.\sigma) + \sum_i^r \langle \omega_i\chi_s, \eta(\alpha_i)\rangle \ ,$$

where ω_i is the relative position $(\sigma(x_i),Q_i)$.

Proposition A.2.1 Let G a semisimple Lie group, Definitions [A.2.2] and [A.2.3] are equivalent.

Proof: It is a consequence of the following equalities:

$$\langle \omega_i\chi_s,\eta(\alpha_i)\rangle = -\langle \eta(\alpha_i),\omega_i\chi_s\rangle = -|\eta(\alpha_i)||\eta(s)|\cos\angle_{\mathrm{Tits}}(\eta(\alpha_i),\omega_i\chi_s) = -\cos\angle_{\mathrm{Tits}}(\eta(\alpha_i),\chi) \ ,$$

where the last equality comes from Definition A.1.3. □

Bibliography

[1] D.N. Akhiezer, *Satake diagrams and real structures on spherical varieties*, International Journal of Mathematics **26**(12) (2015), 13 pp.

[2] N. L. Alling, N. Greenleaf, *Foundations of the theory of Klein surfaces*, Lecture Notes in Mathematics, Springer-Verlag, 1971.

[3] S. Amrutiya, *Real parabolic vector bundles over a real curve*, Proceedings of Mathematical Sciences **124** (2014), 17–30.

[4] M. F. Atiyah, *K-theory and reality*, Q. J. Math. **17**(2) (1966), 367–386.

[5] V. Balaji, I. Biswas and Y. Pandey, *Connections on parahoric torsors over curves* (2017), `arXiv:1702.03623v1`.

[6] V. Balaji and C. S. Seshadri, *Moduli of parahoric G-torsors on a compact Riemann surface*, Journal of Algebraic Geometry **24** (2015), 1–49.

[7] D. Baraglia and L. Schaposnik, *Higgs bundles and (A,B,A)-branes*, Comm. Math. Phys. **331** (2014), 1271–1300.

[8] D. Banfield, *Stable Pairs and principal bundles*, Quart. J. Math. **51** (2000), 417–436.

[9] U. N. Bhosle and A. Ramanathan, *Moduli of Parabolic G-bundles on Curves*, Mathematische Zeitschrift **202** (1989), 161–180.

[10] K. A. Behrend, *Semi-stability of reductive group schemes over curves*, Math. Ann. **301** (1995), 281–305.

[11] I. Biswas, *On the principal bundles with parabolic structure*, J. Math. Kyoto Univ. **43**(2) (2003), 305–332.

[12] I. Biswas and U. N. Bhosle, *Stable real algebraic vector bundles over a Klein bottle*, Trans. Amer. Math. Soc. **360** (2008), 4569–4595.

[13] I. Biswas, L. Calvo, E. Franco and O. García-Prada, *Involutions of the moduli spaces of G-Higgs bundles over elliptic curves*, (2017), `arXiv:1612.08364`.

[14] I. Biswas and O. García-Prada, *Anti-holomorphic involutions of the moduli spaces of Higgs bundles*, Jour. École Poly. **2** (2014), 35–54.

[15] I. Biswas, O. García-Prada and J. Hurtubise, *Pseudo-real principal Higgs bundles on compact Kähler manifolds*, Ann. Inst. Fourier **64** (2014), 2527–2562.

[16] ———, *Pseudo-real principal G-bundles over a real curve*, Journal of the London Mathematical Society **93** (2015), 47–64.

[17] I. Biswas, J. Huisman and J. Hurtubise, *The moduli space of stable vector bundles over a real algebraic curve*, Math. Ann. **347** (2010), 201–233.

[18] I. Biswas and J. Hurtubise, *Principal bundles over a real algebraic curve*, Communications in analysis and geometry **20**(5) (2012), 957–988.

[19] I. Biswas and F. Schaffhauser, *Vector bundles over a real elliptic curve*, Pacific Jour. Math. **283** (2016), 43–62.

[20] O. Biquard, *Fibrés paraboliques stables et connexions singulières plates*, Bull. Soc. Math. France **119** (1991), 231–257.

[21] ———, *Fibrés de Higgs et connexions intégrables: le cas logarithmique (diviseur lisse)*, Ann. Sci. École Norm. Sup. **30**(4) (1997), 41–96.

[22] O. Biquard, O. García-Prada and I. Mundet i Riera, *Parabolic Higgs bundles and representations of the fundamental group of a punctured surface into a real group*, (2016), `arXiv:1510.04207`.

[23] N. Bourbaki, *Lie groups and Lie algebras*, Elements of mathematics [vol. **7**] Part III (ch. 7-9), Springer, (2005).

[24] S. B. Bradlow, *Special metrics and stability for holomorphic bundles with global sections*, J. Diff. Geom. **33** (1991), 169–213.

[25] S. B. Bradlow, O. García-Prada, and I. Mundet i Riera, *Relative Hitchin-Kobayashi correspondences for principal pairs*, Quart. J. Math. **54** (2003), 171–208.

[26] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, GTM 98, Springer–Verlag (1985).

[27] E. Bujalance, J. J. Etayo, J. M. Gamboa and G. Gromadzki, *Automorphism groups of compact bordered Klein surfaces. A combinatorial approach*, Lecture Notes in Mathematics, 1439, Springer-Verlag, (1990).

[28] B. Collier and Q. Li, *Asymptotics of certain families of Higgs bundles in the Hitchin component*, Advances in Mathematics **307** (2017), 488–558.

[29] K. Corlette, *Flat G-bundles with canonical metrics*, J. Diff. Geom. **28** (1988), 361–382.

[30] S. K. Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*, J. Differential Geom., **18** (1983), 269–277.

[31] S. K. Donaldson, *Riemann surfaces*, Oxford Graduate texts in Mathematics **22** (2011).

[32] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. **55**(3) (1987), 127–131.

[33] E. Franco, O. García-Prada and P. E. Newstead, *Higgs bundles over elliptic curves for complex reductive Lie groups*, `arXiv:1310.2168`.

[34] E. Franco, O. García-Prada and P. E. Newstead, *Higgs bundles over elliptic curves for real groups*, `arXiv:1605.04409`.

[35] R. Friedman and J. Morgan, *Holomorphic principal bundles over elliptic curves I*, `arXiv:math/9811130`.

[36] R. Friedman, J. Morgan and E. Witten, *Principal G-bundles over elliptic curves*, Math. Res. Lett. **5** (1998), 97–118.

[37] V. Fock and A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory* Publications Mathématiques de l'IHÉS, **103** (2006), 1–211

[38] W. Fulton and J. Harris, *Representation Theory. A First Course*, Springer (2004).

[39] O. García-Prada, P. B. Gothen, and I. Mundet i Riera, *The Hitchin-Kobayashi correspondence, Higgs pairs and surface group representations*, version 3: 2012, `arXiv:0909.4487`.

[40] O. García-Prada and S. Ramanan, *Involutions on Higgs bundle moduli spaces*, `arXiv:1605.05143`.

[41] O García-Prada and G. Wilkin, *Action of the mapping class group on character varieties and Higgs bundles*, `arXiv:1612.02508`

[42] V. V. Gorbatsevich, A. L. Onishchik, E. B. Vinberg, *Structure of Lie Groups and Lie Algebras.* Translated from the Russian by V. Minachin, [Lie Groups and Lie algebras III], Encyclopaedia Math. Sci., **41**, Springer-Verlag, Berlin, 1991.

[43] B. Gross and J. Harris, *Real algebraic curves*, Ann. scient. Ec. Norm. Sup., **4** (1981), 157–182.

[44] Y. Guivarch, L. Ji, T. C. Taylor, *Compactifications of symmetric spaces*, Progress in Mathematics, Birkhäuser, **156** (1998).

[45] N. J. Hitchin, *Lie groups and TeichmÌler space*, Topology **31** (1992) 449–473.

[46] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **3** (1987) 59–126.

[47] Nan-Kuo Ho, *The Real locus of an involution map on the moduli space of flat connections on a Riemann surface*, Int Math Res Notices **61**(2004), 3263-3285.

[48] Nan-Kuo Ho, G. Wilkin and S. Wu, *Hitchin's equations on a nonorientable manifold*, `arXiv:1211.0746`.

[49] Nan-Kuo Ho and G. Wilkin, *Higgs bundles over a non orientable surface*, `arXiv:1211.0746v1`.

[50] L. Ji, *From symmetric spaces to buildings, curve complexes and outer spaces*, Innov. Incidence Geom. **10** (2009), 33–80.

[51] Nan-Kuo Ho, *The Real locus of an involution map on the moduli space of flat connections on a Riemann surface*, Intern. Math. Res. Not. **61** (2004), 3263–3285.

[52] J. Huisman, *The Equivariant fundamental group, uniformization of real algebraic curves and global complex analytic coordinates on Teichmüller spaces*, Annales de la facultè de sciences de Toulouse, tome 10, n°4, 2001.

[53] M. Kapovich, B. Leeb and J. Millson, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, J. Differential Geom. **81** (2009), 297–354.

[54] F. Klein, *On Riemann's Theory of Algebraic Functions and their Integrals*, Macmillan and Bowes, 1893.

[55] A. W. Knapp, *Lie Groups beyond an Introduction*, second ed., Progress in Mathematics, vol 140, Birkhäuser Boston Inc., Boston, MA, 1996.

[56] M. Liu, F. Schaffhauser, *The Yang-Mills equations over Klein surfaces*, J. Topology **3** (2013), 569-643.

[57] M. S. Narasimhan, C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. Math. (2) **82** (1965), 540-567.

[58] D. Mumford, *Projective invariants of projective structures and applications*, Proc. Intern. Cong. Math., Stockholm (1962), 526-530.

[59] V. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Math. Ann, **248**, (1980), 205-239.

[60] A. Moroianu, *Lectures on Kähler Geometry* Cambridge University Press, 2007.

[61] I. Mundet i Riera, *A Hitchin–Kobayashi correspondence for Kähler fibrations*, J. Reine Angew. Math. **528** (2000), 41–80.

[62] I. Mundet i Riera, *Teoría de Yang-Mills-Higgs para fibraciones simplécticas*, Ph.D. Thesis, 1999.

[63] ———, *A Hilbert-Mumford criterion for polystability in Kähler geometry*, Trans. Am. Math. Soc. **362** (2010), 5169-5187.

[64] ———, *Maximal weights in Kähler Geometry: Flag manifolds and Tits distance* (with an appendix by A. H. W. Schmitt), in: O. García-Prada et. al. (eds.), Vector bundles and complex geometry. Conference on vector bundles in honor of S. Ramanan on the occasion of his 70th birthday, Madrid, Spain, June 16–20, 2008. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics **522** (2010), 113–129.

[65] S. Kobayashi, *Differential geometry of complex vector bundles*, Iwanami Shoten Publishers and Princeton University Press, 1987

[66] G. Pappas, M. Rapoport. *Some questions about $\mathcal{G}$-bundles on curves. Algebraic and arithmetic structures of moduli spaces* Adv. Stud. Pure Math. **58** (2010), 159–171.

[67] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Math. Ann. **213** (1975), 129–152.

[68] A. Ramanathan, *Moduli for principal bundles over algebraic curves:I*, Proc. Indian Acad. Sci. (Math. Sci.) **106** (1996), 301–328.

[69] A. Ramanathan, *Moduli for principal bundles over algebraic curves:I*, Proc. Indian Acad. Sci. (Math. Sci.) **106** (1996), 421–449.

[70] F. Schaffhauser, *Lectures on Klein Surfaces and their fundamental group*, Chapter 2 of Geometry and Quantization of Moduli Spaces, part of the series Advanced Courses in Mathematics, CRM Barcelona, (2016) 67-108

[71] ______, *Real points of coarse moduli schemes of vector bundles on a real algebraic curve*, J. Symplectic Geom. **10** (2012), 503–534.

[72] ______, *Moduli spaces of vector bundles over a Klein surface*, Geom. Dedicata **151** (2012), 187–206.

[73] A. H. W. Schmitt, *Geometric invariant theory and decorated principal bundles*, Zürich Lectures in Advanced Mathematics, Eur. Math. Soc. (2008).

[74] C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Bull. Amer. Math. Soc. **83** (1977), 124–126.

[75] M. Seppala, T. Sorvali, *Geometry of Riemann surfaces and Teichmüller Spaces*, Mathematische Zeitschrift, **201**(2) (1989), 151–165.

[76] C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), 867–918.

[77] ______, *Harmonic bundles on noncompact curves*, J. Amer. Math. Soc. **3** (1990), 713–770.

[78] ______, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. (1992), 5–95.

[79] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I*, Publ. Math., Inst. Hautes Etud. Sci. **79** (1994), 47–129.

[80] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety II*, Publ. Math., Inst. Hautes Etud. Sci. **80** (1995), 5–79.

[81] M. Thaddeus, *Mirror symmetry, Langlands duality and commuting elements of Lie groups*, Int. Math. Res. Not. **22** (2001).

[82] C. H. Taubes, *Differential Geometry: Bundles, Connections, Metrics and Curvature*, Oxford Graduate Texts in Mathematics, (2011).

[83] C. Teleman and C. Woodward, *Parabolic bundles, products of conjugacy classes, and quantum cohomology*, Annales de L'Institut Fourier **3** (2003), 713–748.

[84] S. Wang, *A Narasimhan-Seshadri-Donaldson Correspondence*, Forum Math., **8** (1996) 461–474.

[85] S. Wang, *Moduli spaces over manifolds with involutions*, Mathematische Annalen, **296** (1993) 119–138.

[86] G. Weichold, *Uber symmetrische Riemannsche Flächen und die Periodizitätsmodulen der zugerhörigen Abelschen Normalintegrale erstes Gattung*, Zeitschrift föur Math. und Phys. **28** (1883), 321–351.